International Category Theory Conference 2026

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July 12–18, 2026

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Monday

Monday, July 13, 2026

	Mudd 26		
09:00	**Geoff Cruttwell** The past, present, and future of tangent categories		
10:00	**Brenda Johnson** Distilling Monads		
10:30	Coffee break		
11:00	**Gabriele Lobbia** Constructing Canonical Calculi		
11:30	**Michael Ching** Tangent ∞-Categories and Goodwillie Calculus		
12:00	**Noam Zeilberger** The free bifibration on a functor		
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14:30	**Kimball Strong** Strictification of $(\infty, 1)$-Categories.	**J. Robert Morissette** Dual Equipments	**Sridhar Ramesh** Categorical structures for Gödel incompleteness and Löb's theorem
15:00	**Martina Rovelli** A toolbox for weighted (∞, n)-limits	**Matthew Kukla** Orthogonal Factorization Systems for Double Categories	**Paul Taylor** Transfinite iteration of functors as an extensional reflection
15:30	**Daniel Ramras** Homotopy Pullbacks and Homotopy Groups in CAT	**Ea E T** Exponentiable Virtual Double Categories and Representability of Exponentials	**Joshua Wrigley** Reconstructing the canonical extension in the stable setting
16:00	Coffee break		
16:30	**Daniel Carranza** Homotopy n-types of graphs in discrete homotopy theory	**Diego Manco** Coherence for pseudo commutative 2-monads	**Joseph Hua** Exponentiable morphisms for a clan
16:40	**Eleftherios Chatzitheodoridis** Models for rational $(\infty, 1)$-categories	**Nelson Niu** A Categorical Framework for Coherence Theorems	**João J. Xarez** Monotone-light factorizations, very-well-behaved epireflections and categories of models of sketches
16:50		**Georg Lehner** From Analysis to Stable Homotopy Theory via lax-idempotent Monads	**Yuto Kawase** On the decomposition of a strong epimorphism into regular epimorphisms

Tuesday

Tuesday, July 14, 2026

	Mudd 26		
09:00	**Ieke Moerdijk** Duality for categories of presheaves		
10:00	**Amar Hadzihasanovic** Semi-strictification of (∞, n)-categories		
10:30	Coffee break		
11:00	**David Spivak** Categories by Kan extension		
11:30	**Julie Bergner** Classifying diagrams of double categories with shared isomorphisms		
12:00	**Robert Paré** Lazy Categories		
12:30	Lunch break		
	Krieger 205	Krieger 170	Krieger 180
14:30	**Sam van Gool** Toposes with enough points as categories of étale spaces	**Matthew Di Meglio** Hilbert $*$-categories Where limits in analysis and category theory meet	**Dali Zangurashvili** Effective codescent morphisms in varieties of universal algebras with the amalgamation property
15:00	**Gabriel Saadia** Soberness as idempotent-completeness: towards a formal model theory of virtual ultracategories	**Geoff Vooys** Towards Tangent Structures for Orbifolds by Way of Tangent Categories	**Sergei Burkin** How to relate and generalize nerve, bar-cobar, necklaces, templicial and simplicial objects
15:30	**Marcello Lanfranchi** Local categories: a new framework for partiality	**Khyathi Komalan** Double Categories for Operator Algebras and AQFT	**Omar Dennaoui** Nerve Theorems for Cyclic Operads
16:00	Coffee break		
16:30	**Rubén Maldonado** Double Orthogonal Factorization Systems II	**José Siqueira** Coalgebraic-modal extensions of doctrines	
16:40	**Yuki Imamura** A formal category theoretic approach to the homotopy theory of dg categories	**Yufeng Li** Pointed Univalence	
16:50	**Giuseppe Leoncini** Formal ∞-category theory of relative and simplicial categories, and more general enriched categories.	**Owen Milner** Classifying certain group extensions in HoTT	

Wednesday

Wednesday, July 15, 2026

	Mudd 26
09:00	**Chelsea Walton** Venturing to the Boundary with Braided Module Categories
10:00	**Agustina Czenky** On the classification of modular categories
10:30	Coffee break
11:00	**Luca Mesiti** Monoids, Monoidal Grothendieck Construction and Clifford Semigroups
11:30	**Nicola Gambino** On the 2-category of symmetric 2-rigs
12:00	**Marino Gran** Hopf formulae for cocommutative Hopf algebras
afternoon	**Excursion**

Thursday

Thursday, July 16, 2026

	Mudd 26		
09:00	**Michael Hoefnagel** Coextensivity and arithmeticity in categorical algebra		
10:00	**Eugenia Cheng** The Formal Theory of Iterated Distributive Laws		
10:30	Coffee break		
11:00	**Rory Lucyshyn-Wright** Locally presentable categories over a base, S-sorted limit theories, and cartesian first-order theories		
11:30	**Roy Ferguson** Linearity for monoidal structures		
12:00	**Steve Lack** A Giraud-Conduché condition for T-categories		
12:30	Lunch break		
	Krieger 205	Krieger 170	Krieger 180
14:30	**Rose Kudzman-Blais** Cartesian Linearly Distributive Categories	**Noah D. Ortiz** A Functorial Weak Factorisation System from Path Types	**Ülo Reimaa** Torsion theories as short exact sequences
15:00	**Chun-Yu Lin** Triadic Equivalence of Regular Lawvere Theory	**Jonathan Weinberger** The ∞-category of ∞-categories in simplicial type theory	**Nadja Egner** N-fold groupoids and n-groupoids in regular Mal'tsev categories
15:30	**Claudio Pisani** On the Distributive Law in Cartesian Multicategories	**Maximilian Petrowitsch** Elementary ∞-Toposes from Type Theory	**Manuel Cortés-Izurdiaga** Flatness in Finitely Accessible Additive Categories
16:00	Coffee break		
16:30	**Arghan Dutta** The Morita $(\infty, 2)$-Category of a Monoidal Category as a 2-Complicial Set	**Florian Schwarz** Associated bundles in restriction category world	**Minkyu Kim** Polynomial functors from Lawvere theories
16:40	**Stiéphen Pradal** Weak ∞-categories via fat Delta	**Ruiliang Li** Characterizing Tangent Display Maps via Linear Assignments	**Arnaud Duvieusart** Uniqueness of actions in algebraically coherent categories
16:50	**Johnathon Taylor** Picard Infinity Groupoids with Underlying Globular Sets	**Jean-Baptiste Vienney** Relative differential categories, differential clones and Fermat theories	**David Green** Semisimple Hopf monoidal categories are group theoretical
19:00	**Conference dinner** — Scott Bates Commons Banquet Room		

Friday

Friday, July 17, 2026

	Mudd 26		
09:00	**Susan Niefield** The Double Lives of the Category of Quantales		
10:00	**Evan Patterson** Twisted double functors and their applications		
10:30	Coffee break		
11:00	**Joseph Helfer** Models of set-theory from elementary 2-topoi		
11:30	**Lingyuan Ye** Craig Interpolation for Subgeometric Logics		
12:00	**Davide Trotta** Advances in the Unification of Localic and Realizability Toposes		
12:30	Lunch break		
	Krieger 205	Krieger 170	Krieger 180
14:30	**Giacomo Tendas** Isoregular theories, accessible 2-categories, and free constructions	**James Hefford** Nuclearity and Trace in Monoidal Bicategories and Extended Conformal Field Theories	**Quentin Aristote** Profunctorial algebras
15:00	**Aaron David Fairbanks** Comonads as Spaces	**Joe Moeller** Lyapunov Stability of Coalgebras	**Elena Caviglia** Monadic Approach to Actions of Internal Categories
15:30	**Jason Brown** Completing 2-categories under lax colimits	**Andre Kornell** Constructing the category of quantum graphs	**Victoria Vollmer** On the Category of Graded Monads
16:00	Coffee break		
16:30	**Arne Mertens** Enriched mapping spaces via necklaces	**Léo Schelstraete** Linear higher rewriting and applications to diagrammatic algebras	
16:40	**Nima Rasekh** From Filter Quotient Model Categories to Type Theory	**Florian De Leger** A combinatorial approach to Kontsevich's Swiss cheese conjecture	
16:50	**Jon Funk** Topos polar decomposition	**Christopher Townsend** Why we should pay more attention to Blass' theorem	

Saturday

Saturday, July 18, 2026

	Mudd 26
09:00	**Tobias Fritz** What category theory can do for probability theory
10:00	**Isaiah B. Hilsenrath** Monoidal Differential Turing Categories
10:30	Coffee break
11:00	**Owen Lynch** An Invitation to Geometric Type Theory
11:30	**Paolo Perrone** Presheaves on Markov Categories and Expectation Values
12:00	**Steve Awodey** Path Types in Algebraic Type Theory

The past, present, and future of tangent categories

Geoff Cruttwell

This will be a survey talk consisting of three parts. In the first part, we'll look at what a tangent category is, and how the concept has developed. In the second part, we'll give an overview of some of the key ideas and definitions in tangent category theory, and how they give new perspectives on aspects of differential geometry. In the third part, we'll look at avenues for future work in this area.

What category theory can do for probability theory

Tobias Fritz

The theory of Markov categories offers a synthetic approach to probability theory and statistics, in which measure-theoretic reasoning can often be replaced or reorganized by categorical arguments. In recent years, this perspective has led to purely categorical proofs of classical results in probability and statistics. In this talk, I will focus on another aspect of this story: how the categorical approach provides fresh insights into the nature of probability, its axiomatics, and its distinction from other theories of uncertainty. Particular highlights include simple and intuitive axioms for information flow, and a recent axiomatization of the formation of empirical distributions from infinite sequences of outcomes.

Coextensivity and arithmeticity in categorical algebra

Michael Hoefnagel

Several categorical-algebraic contexts possess well-behaved "measures of abelianness", and therefore naturally produce notions that maximise or minimise such measures. The categories of abelian groups and Boolean rings, for instance, illustrate this extremism with respect to internal notions of *abelian object*: in the former every object is abelian, while in the latter only the terminal object is. Mal'tsev categories have a well-behaved theory of centrality of equivalence relations [2], through which *naturally Mal'tsev categories* [8] and *(proto)arithmetical categories* [9, 1] emerge, in a suitable way, as the centrality maximising and minimising extremes, respectively. From the universal-algebraic point of view, arithmetical algebraic categories are precisely those which are Mal'tsev and congruence distributive, and in other contexts congruence distributivity appears at the opposite-of-abelian end of the spectrum.

The aim of this talk is to show how several topics associated with arithmetical categories and congruence distributive varieties can be approached from the theory of *(co)extensive* categories [3]. For instance, using a notion of *coextensive morphism* developed in [5, 7], we have that a semi-abelian category $\mathcal{C}$ is arithmetical if and only if every product projection in $\mathcal{C}$ is coextensive. A Barr-exact Mal'tsev category is arithmetical if and only if every product projection in each fibre of its fibration of points $\pi_{\mathcal{C}} : \mathrm{Pt}(\mathcal{C}) \to \mathcal{C}$ is coextensive. Outside any particular categorical context, universal-algebraic properties such as the strict refinement property [4], or the *Fraser-Horn property*, may be formulated categorically as assertions that certain canonical classes of morphisms in the algebraic category are coextensive.

The defining exactness properties of Mal'tsev and arithmetical categories may be presented as *matrix properties* in the sense of Z. Janelidze. The final part of this talk presents results from a joint work [6] together with P.-A. Jacqmin and Z. Janelidze which classifies finitely complete categories according to their matrix properties. One such result is that among all non-trivial matrix properties of regular categories, the matrix property corresponding to arithmetical categories is the strongest.

[1] D. Bourn, *A categorical genealogy for the congruence distributive property*, Theory Appl. Categ. **8** (2001), no. 14, 391–407.

[2] D. Bourn and M. Gran, *Centrality and connectors in Mal'tsev categories*, Algebra Universalis **48** (2002), no. 3, 309–331.

[3] A. Carboni, S. Lack and R. F. C. Walters, *Introduction to extensive and distributive categories*, J. Pure Appl. Algebra **84** (1993), no. 2, 145–158.

[4] C. C. Chang, B. Jónsson and A. Tarski, *Refinement properties for relational structures*, Fund. Math. **55** (1964), no. 3, 249–281.

[5] M. Hoefnagel, *$\mathcal{M}$-coextensive objects and the strict refinement property*, J. Pure Appl. Algebra **224** (2020), no. 10, 106381.

[6] M. Hoefnagel, P.-A. Jacqmin and Z. Janelidze, *The matrix taxonomy of finitely complete categories*, Theory Appl. Categ. **38** (2022), 737–790.

[7] M. Hoefnagel and E. Theart, *On extensivity and coextensivity of morphisms*, Theory Appl. Categ. **44** (2025), no. 21, 617–642.

[8] P. T. Johnstone, *Affine categories and naturally Mal'cev categories*, J. Pure Appl. Algebra **61** (1989), no. 3, 251–256.

[9] M. C. Pedicchio, *Arithmetical categories and commutator theory*, Appl. Categ. Structures **4** (1996), no. 2–3, 297–305.

Duality for categories of presheaves

Ieke Moerdijk

Joint work with: Eric Hoffbeck

I will give examples of small categories for which there is a bar-cobar duality between the category of presheaves of chain complexes on that small category and the category of copresheaves of such. Several up-to-homotopy (co-)algebraic structures can be defined terms of such (co-)presheaves equipped with extra structure, and this extra structure is respected by the duality. The duality specializes to bar-cobar duality theorems for infinity-algebras, infinity-operads and algebras over the latter, and equivariant versions of these.

The Double Lives of the Category of Quantales

Susan Niefield

It is well known that if M is a module over a commutative ring R, then the endofunctor $- \otimes_R M$ has a left adjoint if and only if M is a finitely generated and projective. Following their ring/quantale analogy, Joyal and Tierney [1] considered monoids in the symmetric monoidal category of complete lattices and sup-preserving maps, later called quantales, and proved the above projectivity characterization for modules over a commutative quantale, without the finiteness condition. In a paper with Wood [2], we proved a general theorem characterizing the existence of a left adjoint to the functor $- \otimes_R M$ on the category of modules over a commutative monoid R in a suitable symmetric monoidal categories $\mathcal{V}$, and obtained corollaries for rings and quantales.

In [4], Paré defined adjoints and Cauchy completeness for double categories and, considered the double category $\mathbb{R}\mathsf{ing}$ of commutative rings, homomorphisms, and bimodules. There, an (R, Q)-bimodule M has a right adjoint if and only if it is finitely generated and projective as an R-module, and so there are no non-trivial Cauchy complete rings. To overcome this deficiency, he replaced the homomorphisms in $\mathbb{R}\mathsf{ing}$ by maps he called amplimorphisms, and obtained a double category $\mathbb{A}\mathsf{mpli}$ of rings in which every object is Cauchy complete. Subsequently [3], we incorporated Paré's adjoint bimodule result into a version of the 2017 theorem with Wood, which we then applied to rings and quantales. However, the proofs of the latter were separate due to the finiteness condition for rings but not quantales, and we did not construct an $\mathbb{A}\mathsf{mpli}$-like double category for quantales.

After recalling the necessary background, we present three double categories whose objects are quantales. The first is strict, the second is pseudo, and the third is a double bicategory, in the sense of Verity [5]. The strict double category is Cauchy, i.e., every object is Cauchy complete. The pseudo one is not, but this is corrected using a Kleisi-like construction. To do so, we assume additional conditions on $\mathcal{V}$ and introduce a general notion of projective module over a monoid in $\mathcal{V}$ which, when added to the 2025 theorem, gives a single theorem which applies simultaneously to rings and quantales. Finally, we define morphisms of quantales like Paré's amplimorphisms of rings, and construct a Cauchy double bicategory of quantales.

[1] A. Joyal and M. Tierney, *An Extension of the Galois Theory of Grothendieck*, Amer. Math. Soc. Memoirs 309, (1984).

[2] S. Niefield and R. Wood, Coexponentiability and projectivity: rigs, rings, and quantales, TAC 32 (2017), 1222–1228.

[3] S. Niefield, Cauchy completeness and adjoints in double categories, TAC 43 (2025), 9–22.

[4] R. Paré, Morphisms of rings, Outstanding Contributions to Logic 20, Springer (2021), 271–298.

[5] D. Verity, Enriched Categories, *Internal Categories and Change of Base*, Ph.D. Thesis, Cambridge University, 1992. Republished as: Reprints in Theory and Applications of Categories 20 (2011), 1–266.

Venturing to the Boundary with Braided Module Categories

Chelsea Walton

Joint work with: Monique Müller, Robert Laugwitz, Harshit Yadav, and Milen Yakimov

While the relationship between braided monoidal categories and quantum groups has successfully unlocked deep connections in knot theory and the Yang-Baxter equation, extending this framework to boundary phenomena requires shifting our focus to braided module categories. This talk explores this boundary setting first by introducing the "reflective center" of a module category, showing how it naturally produces solutions to the boundary Yang-Baxter equation (also known as the quantum reflection equation). We will then examine when braided module categories are "nondegenerate", which is analogous to the notion for braided finite tensor categories used to construct 3d-TQFTs. Finally, we will present concrete connections to braid groups of type BC and D, mirroring the classic connections to braid groups of type A.

Profunctorial algebras

Quentin Aristote

Joint work with: Umberto Tarantino

The aim of this talk is to present a 2-dimensional version of Barr's landmark *Relational algebras* paper [1]. In this paper, Barr first noticed that, viewing relations $R \subseteq X \times Y$ as spans of their projections $X \leftarrow R \rightarrow Y$, there is a natural way to extend the action of a functor $F\colon \mathbf{Set} \rightarrow \mathbf{Set}$ to $\mathbf{Rel}$ — namely, by having F act on the two projections. Such an extension preserves the natural order of relations; it also preserves their composition precisely when F preserves *weak pullbacks*, in which case F extends to a 2-functor $\underline{F}\colon \mathbf{Rel} \rightarrow \mathbf{Rel}$. Similarly, a monad structure on F extends to that of a lax monad, and it can be characterized when this extension is strict. Barr's leading application of this result was to characterize the category $\mathbf{Top}$ of topological spaces as the *relational algebras* for the ultrafilter monad β, that is, the *lax algebras* of its extension $\underline{\beta}\colon \mathbf{Rel} \rightarrow \mathbf{Rel}$.

We extend both results to the setting of *bicategories*. Building on [3, 4, 5], relations in a bicategory K can be identified with *two-sided discrete fibrations*, which determine a bicategory $\mathbf{TSDFib}(\mathsf{K})$ if K satisfies some conditions akin to 1-dimensional *regularity*. Our first main result, formulated in terms of exactness *à la* Guitart [6], reads as follows.

Theorem. A pseudomonad $\langle \mathsf{T}, \eta, \mu \rangle$ on a regular bicategory K extends to a pseudomonad $\langle \underline{\mathsf{T}}, \underline{\eta}, \underline{\mu} \rangle$ on $\mathbf{TSDFib}(\mathsf{K})$ if and only if:

1. $\mathsf{T}\colon \mathsf{K} \rightarrow \mathsf{K}$ preserves exact squares, and
2. the naturality squares of its unit $\eta\colon \mathrm{id} \Rightarrow \mathsf{T}$ and multiplication $\mu\colon \mathsf{T}^2 \Rightarrow \mathsf{T}$ are exact.

We then focus on the case of $\mathbf{CAT}$, the 2-category of (locally small) categories, so that $\mathbf{TSDFib}(\mathsf{K})$ can be identified with $\mathbf{PROF}$, the bicategory of categories and *small profunctors*. In his talk at CT24, Rosolini introduced the *ultracompletion* pseudomonad on $\mathbf{CAT}$ to study Makkai's *ultracategories* [7] as its pseudoalgebras. In this talk, we will make a *profunctorial* version of ultracategories emerge as algebras for the extension of the ultracompletion pseudomonad to $\mathbf{PROF}$: the resulting notion will recover *ultraconvergence spaces*, the categorification of topological spaces recently introduced in [8, 9] to extend Makkai's *Stone duality for first order logic* [7] to geometric logic.

[1] M. Barr, *Relational algebras*, Reports of the Midwest Category Seminar IV. Ed. by S. Mac Lane et al., Springer (1970), 39–55.

[2] Q. Aristote and U. Tarantino, *Profunctorial algebras*, in preparation, 2026.

[3] A. Carboni, S. Johnson, R. Street, and D. Verity, *Modulated bicategories*, Journal of Pure and Applied Algebra, vol. 94, no. 3, pp. 229–282, 1994.

[4] R. Street, *Fibrations in bicategories*, Cahiers de topologie et géométrie différentielle, vol. 21, no. 2, pp. 111–160, 1980.

[5] F. Loregian and E. Riehl, *Categorical notions of fibration*, Expositiones Mathematicae, vol. 38, no. 4, pp. 496–514, 2020.

[6] R. Guitart, *Relations et carrés exacts*, Annales des sciences mathématiques du Québec, vol. 4, no. 2, pp. 103–125, 1980.

[7] M. Makkai, *Stone duality for first order logic*, Advances in Mathematics **65**.2 (1987), 97–170.

[8] G. Saadia, *Extending conceptual completeness via virtual ultracategories*, preprint arXiv:2506.23935, 2025.

[9] S. van Gool, J. Marquès and U. Tarantino, *Toposes with enough points as categories of étale spaces*, preprint arXiv:2508.09604, 2025.

Path Types in Algebraic Type Theory

Steve Awodey

Joint work with: Joseph Hua

We propose a new approach to the semantics of identity types in intensional Martin-Löf type theory. The setting is quite general, assuming only a category with finite limits and an exponentiable *interval* $1 \rightrightarrows I$. It therefore applies to many different cases such as clans [1], model categories [2], path categories [3], categories with families [4], and natural models [5]. This approach is being used in the HoTTLean project [6], where it has already been formalized in Lean.

The specification of *extensional* identity types in natural models paralleled that of the other type formers Σ and Π, but the treatment of the *intensional* case was less uniform. The latter was updated in [7] to a cleaner account suggested by R. Garner using polynomials. Here that approach is improved further by employing an interval to give a specification entirely analogous to that of the other type formers. We require namely a pullback diagram of the following form for *path types*, where $\dot{\mathsf{T}} \to \mathsf{T}$ is the natural model.

$$\begin{array}{ccc} \dot{\mathsf{T}}^I & \longrightarrow & \dot{\mathsf{T}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \dot{\mathsf{T}} \times_{\mathsf{T}} \dot{\mathsf{T}} & \longrightarrow & \mathsf{T} \end{array} \tag{1}$$

The interval is also used to specify a (Hurewicz) fibration structure on $\dot{\mathsf{T}} \to \mathsf{T}$. The combination of these two conditions already suffices to validate the usual rules for intensional identity types. The presence of an interval moreover relates the new approach to that of cubical type theory [8]. Indeed, the category of types is thereby enriched in cubical sets, and the type families classified by $\dot{\mathsf{T}} \to \mathsf{T}$ are then necessarily cubical Kan fibrations in the sense of [9].

Many familiar models of type theory are subsumed as examples, including locally cartesian closed categories (the 0-truncated case), the usual groupoid model (the 1-truncated case), and certain monoidal model categories, such as simplicial and cubical sets (the untruncated case).

[1] André Joyal. Notes on clans and tribes. *arXiv:1710.10238*, 2017.

[2] Michael Shulman. All $(\infty, 1)$-toposes have strict univalent universes. *arXiv:1904.07004*, 2019.

[3] Benno van den Berg and Ieke Moerdijk. Exact completion of path categories and algebraic set theory: Part I: Exact completion of path categories. *Journal of Pure and Applied Algebra*, 222(10), 2018.

[4] Simon Castellan, Pierre Clairambault, and Peter Dybjer. Categories with families: Unityped, simply typed, and dependently typed. In *Joachim Lambek: The Interplay of Mathematics, Logic, and Linguistics.* Springer, 2021.

[5] Steve Awodey. Natural models of homotopy type theory. *Mathematical Structures in Computer Science*, 28(2), 2016.

[6] Joseph Hua, Steve Awodey, Mario Carneiro, Sina Hazratpour, Wojciech Nawrocki, Spencer Woolfson, and Yiming Xu. HoTTLean: Formalizing the meta-theory of HoTT in Lean. *TYPES 2025.*

[7] Steve Awodey. Algebraic type theory. *International Category Theory Conference CT2024.* Santiago de Compostela, Spain, 2024.

[8] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical type theory: A constructive interpretation of the univalence axiom. *TYPES 2015.*

[9] Steve Awodey. *Cartesian cubical model categories.* Volume 2385 of *Lecture Notes in Mathematics.* Springer, 2026.

Classifying diagrams of double categories with shared isomorphisms

Julie Bergner

Joint work with: Brandon Shapiro, Inna Zakharevich

Abstract: The classifying diagram construction, originally due to Rezk [4], provides a way to produce a complete Segal space from an ordinary category, in a way that refines the usual nerve construction. In particular, two categories are equivalent if and only if their classifying diagrams are levelwise equivalent as simplicial spaces. The key feature of the classifying diagram, compared to the ordinary nerve, is that it provides a way to distinguish isomorphisms from ordinary morphisms in the category.

To make sense of an analogous classifying diagram for a double category, it turns out that we need the horizontal and vertical categories to have the same isomorphisms. Making the necessary conditions precise, we arrive at the definition of double categories with shared isomorphisms, which have interesting features in their own right. In joint work with Shapiro and Zakharevich, we identify when a simplicial space is the classifying diagram of a category, and when a bisimplicial space is the classifying diagram of a double category with shared isomorphisms, in terms of various lifting conditions [2].

Our motivation for defining classifying diagrams for double categories was when we surprisingly found them to provide a useful characterization of which pointed stable double categories in the sense of [1] correspond to CGW categories in the sense of [3], thus providing the relationship between two general inputs for algebraic K-theory constructions.

[1] Julia E. Bergner, Angélica M. Osorno, Viktoriya Ozornova, Martina Rovelli, and Claudia I. Scheimbauer, 2-Segal objects and the Waldhausen construction, *Alg. Geom. Topol.* 21 (2021) 1267–1326.

[2] Julia E. Bergner, Brandon T. Shapiro, and Inna Zakharevich, Recognizing CGW categories among pointed stable double Segal spaces, in preparation.

[3] Jonathan Campbell and Inna Zakharevich, Dévissage and localization for the Grothendieck spectrum of varieties, *Adv. Math.* 411 (2022), Paper No. 108710, 80 pp.

[4] Charles Rezk, A model for the homotopy theory of homotopy theory, *Trans. Amer. Math. Soc.* , 353(3), 2001, 973–1007.

Completing 2-categories under lax colimits

Jason Brown

Given a class of colimits, Φ, one can freely add Φ-colimits to a category to obtain its *free completion under Φ-colimits*. Examples include the (small) presheaf category (when Φ = all small colimits), the Fam construction (Φ = coproducts), the Ind-completion (Φ = filtered colimits) and the Karoubi envelope (Φ = splitting of idempotents).

Higher-dimensional categories provide richer notions of colimit for which we can describe corresponding completions. For example, both the *Kleisli completion* for 2-categories and the map sending a bicategory $\mathcal{V}$ to $\mathcal{V}\mathsf{Cat}$ can be viewed as free completions under certain classes of *lax* colimits for 2-dimensional categories, as shown in [2] and [1] respectively.

This talk will provide descriptions first for the completion of a 2-category under the class of *all* small lax colimits, and then for certain subclasses of lax colimits. In particular, we will observe that completing a 2-category under lax colimits of lax functors from small 1-categories yields a construction analogous to the Fam construction for ordinary categories, both in its structure and properties. This completion extends the Kleisli completion (i.e. the completion under lax colimits of lax functors from the terminal category) and is related to enrichment: the objects of the lax-functor lax-colimit completion of the delooping $\mathbb{B}\mathcal{V}$ of a monoidal category $\mathcal{V}$ can be viewed as categories enriched in $\mathsf{Fam}(\mathcal{V})$.

Notable examples include the lax-functor lax-colimit completions of $\mathsf{Cat}_{\cong}$ (with 2-cells natural *isomorphisms*), Grpd and Set, which yield respectively the 2-categories of fibrations Fib, IsoFib and $\mathsf{DiscFib}$, whereas the completion of the arrow category $\mathbf{2}$ yields a certain 2-category of profunctors.

If time permits, we will also observe how free completions under classes of lax colimits naturally extend from 2-functors on $\mathsf{2Cat}$ to $\mathsf{Gray}_{\mathsf{lax}}$-functors, i.e. functors which additionally act on *lax* transformations and modifications between them.

This talk is based on work from my PhD thesis [3] supervised by Richard Garner.

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How to relate and generalize nerve, bar-cobar, necklaces, templicial and simplicial objects

Sergei Burkin

It is known that the category of comonoids in *Sets* is equivalent to the category *Sets* itself. This can be rephrased as follows: the inclusion of the initial operad, whose only operation is the identity, into the operad uAs of monoids induces equivalence of corresponding categories of coalgebras.

We will describe a new zig-zag $sOp_{uAs//uAs}(1) \to sOp_{uAs//uAs} \leftarrow Tw''(uAs)$ of inclusions of operads that induces equivalence of corresponding categories of coalgebras, where on the left side the coalgebras are simplicial sets, and on the right side the coalgebras are non-symmetric cooperads. The functor from one side to the other has already appeared in [1, Example 3.6.5]. In general this equivalence breaks in enriched setting, however the right side is still quite interesting, being related to the necklace category of Dugger and Spivak and to templicial objects.

The operad uAs in the above zig-zag can be replaced by any other operad P. The constructions $sOp_{P//P}$ and $Tw''(P)$ are both reasonable generalizations of the notion of twisted arrow category of a category to operads, and have appeared respectively in [2, 3] and in [4]. As a particular case we recover the category of dendroidal necklaces and some of the key categories from [5].

Additionally, we show that both constructions $sOp_{P//P}$ and $Tw''(P)$ fit into a larger picture that also involves the root functor of [6], which is a generalization of the last vertex map to dendroidal setting.

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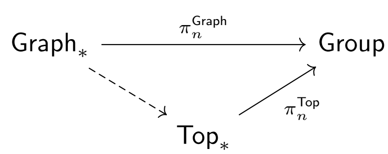
Homotopy n-types of graphs in discrete homotopy theory

Daniel Carranza

Joint work with: Mark Behrens, Chris Kapulkin

Discrete homotopy theory, introduced around 20 years ago by H. Barcelo and collaborators [1] building on the work of Atkin from the mid-seventies, is a homotopy theory of (simple) graphs. The theory reimagines the foundations of algebraic topology in the category of graphs, and the resulting homotopy invariants encode more combinatorial information within a graph. These invariants have found applications in matroid theory, hyperplane arrangements, time series analysis, and most recently in topological data analysis, where it provides a more noise-resistant alternative to the usual Vietoris-Rips construction.

Recently, tools from category theory and abstract homotopy theory have been successfully used to prove new results in the field, including a resolution [2] of the Babson–Barcelo–de Longueville–Laubenbacher conjecture [3] on realizing discrete homotopy groups of graphs. That is, the discrete homotopy groups functor $\pi_n^{\mathrm{Graph}} \colon \mathsf{Graph}_* \to \mathsf{Group}$ factors through the topological homotopy groups functor as in the diagram:



The functor $\mathsf{Graph} \to \mathsf{Top}$ itself factors through the category cSet of cubical sets. Similar to simplicial sets, cubical sets are a model for the homotopy theory of spaces, and are defined as the category of presheaves over the *box category*. The resulting functor $\mathsf{Graph} \to \mathsf{cSet}$ is known as the *nerve functor*. The conjecture was resolved by using the model structure on the category of cubical sets together with the closed symmetric monoidal product given by the *geometric product* of cubical sets; thus the discrete homotopy groups of a graph can be recovered as the topological homotopy groups of the geometric realization of its nerve.

A major problem in the field is to determine whether the nerve functor is an equivalence of ∞-categories between the category of graphs, localized at π_*^{Graph}-isomorphisms, and the ∞-category of spaces. A positive resoluton to this conjecture would establish a "dictionary" by which results in algebraic topology can be translated to results in the homotopy theory of graphs. This would resolve numerous other open problems, including discrete analogues of the Blakers–Massey theorem, the Mayer–Vietoris long exact sequence, and Brown representability.

In this talk, I will report on joint work with Mark Behrens and Chris Kapulkin making progress towards the conjecture by showing the nerve functor is essentially surjective after localizing at n-equivalences. That is, for a fixed n, the homotopy groups up to dimension n of any topological space can be recovered as the discrete homotopy groups of some graph. Thus, our construction yields a potential inverse to the nerve functor up to homotopy. Our proof combines cubical homotopy theory, Reedy theory, and explicit combinatorics on graphs. As an added benefit, our methods allow us to compute previously-unknown discrete homotopy groups.

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Monadic Approach to Actions of Internal Categories

Elena Caviglia

Joint work with: Sophie Marques, Luca Mesiti

Group actions are one of the most fundamental basic tools in algebra and geometry. An important categorical result is that sets equipped with an action of a fixed group can be captured as algebras for a monad on Set. This monad, called the action monad or also the writer monad, has had abundant applications in both mathematics and computer science. Notably, one of the outcomes of the monadicity result for group actions is the consequent notion of internal action of a group object in a category $\mathcal{C}$. Indeed, this notion is given by generalizing the action monad to a monad on $\mathcal{C}$.

More recently, George Janelidze and Walter Tholen introduced a notion of action of an internal category in a category $\mathcal{C}$ on an object of $\mathcal{C}$. This was motivated by important applications to descent theory.

In this talk, we prove a monadicity result for these actions of an internal category. We reach this by generalizing the action monad for internal group actions to a monad on the slice category over the object of objects of the internal category. The classical action monad for a group object is recovered when viewing the group object as an internal category with object of objects given by the terminal object.

As an outcome of this monadicity result, we show that the process of associating to an internal category its category of actions is functorial, as it corresponds to a morphism of monads. This is an important ingredient that we needed for applications to categorical Galois theory.

Moreover, we completely characterize which monads arise as monads of actions of internal categories. Such monads have as base category a slice category $\mathcal{C}/X$, and they are equipped with a comorphism of monads towards the identity monad. This in particular implies that the category $\mathcal{C}$ is embedded in the category of algebras. In the case of actions of a group object G, the presence of such embedding corresponds to the fact that every object of the category can be equipped with a trivial action of G given by the projection. Surprisingly, this property turns out to be fundamental in identifying the monads of internal actions.

Finally, we show some applications of this work to categorical Galois theory as well as algebraic geometry, which motivated us to develop these results.

Models for rational $(\infty, 1)$-categories

Eleftherios Chatzitheodoridis

The study of $(\infty, 1)$-categories is intended to understand categories with a notion of homotopy between their morphisms. Namely, an $(\infty, 1)$-*category* is a category enriched in spaces, possibly weakly. This enrichment yields k-morphisms for all k given by homotopies. However, all morphisms in dimension higher than 1 are invertible up to homotopy because homotopies can be traveled in reverse.

Our understanding of $(\infty, 1)$-categories has been advanced thanks to the development of various models for $(\infty, 1)$-categories, that is, mathematical objects that exhibit the structure of an $(\infty, 1)$-category. Two such models are *complete Segal spaces*, as introduced by Rezk in [7], and *Segal categories*, as developed from the homotopical perspective by Bergner in [1].

In [3], we develop an analog of Bergner and Rezk's work for rational homotopy theory. We introduce *rational* $(\infty, 1)$-*categories*, which are $(\infty, 1)$-categories enriched in spaces whose higher homotopy groups are rational vector spaces. Then, we produce two models for rational $(\infty, 1)$-categories, *rational complete Segal spaces* and *rational Segal categories*.

Our argument is not exclusive to rational homotopy theory; it works for enrichment in general localizations of spaces. For example, in the chromatic homotopy-theoretic framework developed by Heuts in [4], our work yields two equivalent models for $(\infty, 1)$-categories enriched in v_n-periodic spaces for a non-negative integer n, with the case $n = 0$ being that of rational $(\infty, 1)$-categories.

Lastly, we present the future directions of our project on developing rational homotopy-theoretic analogs of other models for $(\infty, 1)$-categories. Such models are *quasi-categories*, as investigated by Joyal in [5] and Lurie in [6], as well as *simplicial categories*, as studied by Bergner in [2]. We discuss our ongoing work on giving localized analogs of these two models for $(\infty, 1)$-categories.

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The Formal Theory of Iterated Distributive Laws

Eugenia Cheng

Joint work with: Alexander S. Corner

In this talk we will use the Formal Theory of Monads [4] to shed more light on the theory of Iterated Distributive Laws [2].

The standard theory of distributive laws [1] gives a way to compose two monads on the same category, or equivalently, to lift one monad to the category of algebras of the other in such a way that the algebras for the lifted monad and those for the composite monad coincide. This gives two points of view on the same composite algebraic structure. A motivating example gives two constructions of 2-categories: on the one hand we can use two monads on 2-graphs, one for vertical composition and one for horizontal composition, and on the other hand we can use the free category monad on 1-graphs, and then the free Cat-enriched category monad on Cat-graphs; the resulting structures coincide.

Street's classic paper [4] constructs, for any 2-category K, a 2-category $\mathsf{Mnd}(K)$ of monads on K, monad functors, and monad transformations. The paper ends with several enticing results about distributive laws, including that distributive laws are the 0-cells of $\mathsf{Mnd}(\mathsf{Mnd}(K))$. Furthermore, Mnd is a monad on $\mathsf{2\text{-}Cat}$, whose composition takes a distributive law and returns the composite monad.

In [2] the first author extended the theory of distributive laws to be able to compose n monads with coherent distributive laws between them. In this talk we will further extend this to include iterated lifts of the monads and distributive laws between them. While this can be proved directly we find it more interesting to prove it as a corollary of a small extension of [4]. There, the existence of Eilenberg–Moore objects for monads on a given K is expressed via a 2-functor $\mathsf{Alg}\colon \mathsf{Mnd}(K) \longrightarrow K$ picking out the Eilenberg–Moore object for each monad in K. We prove that in fact Alg exhibits such K as a pseudo-algebra for Mnd regarded as a 2-monad on the 2-category $\mathsf{2\text{-}Cat}$. The associativity isomorphism "is" Beck's standard result about algebras via distributive laws. This result is so natural that we suspect it might be considered "folklore"; however we have not seen or heard it mentioned. The proof is long but routine.

Our result about iterated distributive laws follows: given a distributive series of n monads, we can lift the first $n-1$ monads to the category of algebras for the last monad (by [1]), but moreover all the distributive laws lift too, so the process can be iterated, giving an $n-1$ times iterated lift of the first monad in the series. In the case of n-categories, this gives a more abstract proof that the n-categories constructed by composing the individual monads for composition in each direction coincide with the n-categories constructed by iterated enrichment.

This extension of [2] gives us two main benefits that we use in our work. The first is an extension to 2 dimensions, where we look at (strict) distributive laws between 2-monads. The 2-monads and the distributive laws remain strict so the basic definitions do not have any subtlety, but we can extend the lifting theorem to the 2-categories of pseudo-algebras; this is crucial for our continued work on semi-strict n-categories [3].

Our further application is for a generalised Eckmann–Hilton argument on n-degenerate $(n+1)$-categories. Our re-framing of [2] enables us to give a satisfying abstract account of different expressions of the Eckmann–Hilton argument: via iterated internalisation, or via multiple multiplicative structures and interchange. The equivalence of those structures is usually proved by direct calculation, but we prefer the abstract argument provided by the present work.

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Tangent ∞-Categories and Goodwillie Calculus

Michael Ching

Joint work with: Kristine Bauer, Matthew Burke

Following Rosický [Ros84], Cockett and Cruttwell introduced in [CC14] a notion of *tangent category* to axiomatize certain categorical properties of the tangent bundle construction on a smooth manifold. Examples of tangent categories appear in many contexts including commutative algebra and algebraic geometry [CL23], operad theory [ILL24], logic, category theory, and, of course, differential geometry (ordinary and synthetic). With Bauer and Burke [BBC21], we recently added homotopy theory to this list by extending Cockett and Cruttwell's definition to ∞-categories in the sense of Lurie, constructing an example of a tangent ∞-category that encodes Goodwillie's functor calculus [Goo03].

Our definition depends on a certain symmetric monoidal ∞-category $\mathbb{W}\mathrm{eil}_\infty$ of ∞-*Weil-algebras*. That ∞-category is related to, though not the same as, the category of Weil-algebras introduced by Leung in [Leu17] to give an alternative characterization of tangent structure. In this talk, I will focus on the ∞-category $\mathbb{W}\mathrm{eil}_\infty$, giving two different descriptions, one in terms of partial commutative monoids and one in terms of E_∞-semirings. The part that is new is the proof that these two descriptions are equivalent, which provides us with additional examples of tangent ∞-categories of E_∞-ring spectra. Our work also determines a notion of tangent (∞-)bicategory, which we propose as a natural way to extend Cockett and Cruttwell's work to a bicategorical setting.

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Flatness in Finitely Accessible Additive Categories

Manuel Cortés-Izurdiaga

The notion of κ-accessible category, where κ is an infinite regular cardinal, was introduced by Lair in [5] under the name of *catégorie modelable*, as an extension of the κ-presentable categories previously studied by Gabriel and Ulmer in [4]. The term accessible was first used by Makkai and Paré in [6].

In the talk, we will be interested in finitely accessible (i. e., $\aleph_0$-accessible) additive categories $\mathbf{C}$. These categories have all directed colimits and a set of finitely presented objects such that every object in the category is a directed colimit of a directed system of such objects.

Even in the non-additive case, there is a notion of purity in such categories, that allows, in the additive setting, to define an exact structure in the sense of Quillen [8]. The admissible monomorphisms and epimorphisms (also called inflations and deflations, respectively) in this exact category are the pure monomorphisms and pure epimorphisms, respectively. Of course, not every epimorphism $f : A \to F$ is a pure epimorphism, but there might exist objects F satisfying the property that every epimorphism ending at F is pure. These objects are called *flat*. If $\mathbf{C}$ is a module category (over a non-commutative ring) or, more generally, a finitely accessible Grothendieck category, this notion agrees with the classical notion of flat object first considered by Stenström [9] it this categorical setting.

In the recent work [2], Cuadra and Simson studied flat objects in the category of comodules over a coalgebra and they stated some open problems for a finitely accessible Grothendieck category $\mathbf{G}$: (a) Give a characterization of those categories $\mathbf{G}$ that have enough *flat objects* (i. e., every object in the category is an epimorphic image of a flat object); (b) If there are enough flat objects in $\mathbf{G}$, are there enough projectives?

In the talk, which is based on the paper [1], we will show how (a) can be solved using the representation theorem of finitely accessible additive categories via functor categories given by Crawley-Boevey [3]. Regarding (b), we will give a criterion to determine when the finitely accessible additive category $\mathbf{C}$ has enough projective objects, provided that it has enough flat objects. This will allow us to give an affirmative answer to (b) in some particular situations.

Recently, Martini, Parra, Saorín and Virili [7] have shown that question (b) has a negative answer. Let us point out that our results are valid for finitely accessible additive categories satisfying certain mild conditions, which are much weaker than being Grothendieck.

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On the classification of modular categories

Agustina Czenky

Joint work with: Akshaya Chakravarthy, William Gvozdjak, Julia Plavnik (separate works)

Title: On the classification of modular categories

Abstract: A modular tensor category (MTC) is a fusion category with braiding and ribbon structures, which satisfy a non-degeneracy condition. They are of interest for a variety of mathematical subjects, such as topological quantum field theory, representation theory of quantum groups, von Neumann algebras, conformal field theory and vertex operator algebras. In particular, modular tensor categories sit at the crossroads of quantum algebra and condensed-matter physics, providing an algebraic model for anyon systems arising from topological phases of matter, which are currently viewed as potential hardware for fault-tolerant quantum computing. Although these categories play a central role in both mathematics and physics, their overall landscape remains only partially understood, making their classification a rich and challenging problem.

A major breakthrough showed that for any fixed rank, defined as the number of isomorphism classes of simple objects, there exist only finitely many MTCs [BNRW]. This finiteness result has led to a systematic classification program by rank, with complete or partial results now known in small ranks and for various restricted classes. Recently, [NRW] presented a classification of modular data up to rank 11.

One particularly tractable and well-motivated subclass consists of integral MTCs, where all simple objects have integer Frobenius–Perron dimension; these categories are in correspondence with the categories of representations of modular finite-dimensional semisimple quasi-Hopf algebras.

Within the integral setting, MTCs of odd Frobenius–Perron dimension form a distinguished family. These categories are maximally non-self-dual, a property that significantly reduces the complexity of classification and leads to strong structural constraints. In joint work, we show that all such categories of ranks 13 and 15 are pointed, and that those of ranks 19–23 are necessarily pointed as well, completing their classification. More broadly, our methods yield structural results for odd-dimensional MTCs in substantially higher ranks, showing that in many cases such categories must be either pointed or perfect [CP, CGP]. On the other hand, it turns out that odd-dimensional MTCs and MTCs whose Frobenius–Perron dimension is congruent to 2 modulo 4 share many properties. In [CCP], we use these similarities to extend the classification work of such categories. By contrast, MTCs whose Frobenius–Perron dimension are divisible by 4 are still relatively unexplored.

In this talk, I will give an overview of the classification program for modular categories, focusing on key invariants such as rank, quantum dimensions, and modular data, and on the construction techniques that allow new examples to be built from known ones.

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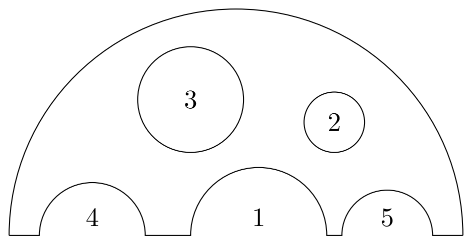
A combinatorial approach to Kontsevich's Swiss cheese conjecture

Florian De Leger

The *Swiss cheese operad* was introduced and studied by Voronov [3]. It is a topological two-coloured operad, one colour corresponding to full disks and the other to half disks. The space of operations with target colour *full disk* is the space of ordered configuration of non-overlapping little disks inside the unit disk. The space of operations with target colour *half disk* is the space of ordered configuration of non-overlapping little disks or little half disks inside the unit half disk:



The operadic composition is given by plugging disks or half disks then reordering.

We will explain how to produce a new coloured operad $\mathrm{SC}(\mathcal{P})$ from any coloured operad $\mathcal{P}$. We will then present the main result of [1] which states that if $\mathcal{P}$ is the little intervals operad, then $\mathrm{SC}(\mathcal{P})$ is equivalent to the Swiss cheese operad. This gives us a weak version (without the universal property) of Kontsevich's Swiss cheese conjecture [2].

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Nerve Theorems for Cyclic Operads

Omar Dennaoui

The classical nerve theorem establishes that categories can be characterized as simplicial sets satisfying the Segal condition, providing a bridge between categorical and simplicial perspectives. Given a simplicial set X, it is the nerve of a category if and only if the following Segal map

$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

is an isomorphism for all $n \geq 2$. This characterization has proven fundamental in higher category theory and has been extended to various algebraic structures. We establish analogous nerve theorems for (augmented) cyclic multicategories and cyclic operads [2, 4], allowing one to profitably view these structures as presheaves on a category of (planar) trees. Here "cyclic" refers to a rotational symmetry on operations that blurs the distinction between inputs and outputs. Cyclic multicategories and operads play important roles in multivariable adjunctions [2], cyclic homology, graph complexes, and topological field theories, among other applications.

We employ the general frameworks for abstract nerve theorems developed by Leinster and Weber. Specifically, we apply the machinery of monads with arities of Berger, Melliès, and Weber [1], it provides a systematic method for computing minimal dense generators of categories of algebras over strongly cartesian monads. We instantiate this for monads on the category of cyclic multigraphs to establish the theorems; in the case of the free cyclic operad monad, this was anticipated in Elliott's thesis [3].

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Hilbert $*$-categories

Where limits in analysis and category theory meet

Matthew Di Meglio

Joint work with: Chris Heunen

This talk will introduce *Hilbert $*$-categories* [1, 2]—an abstraction capturing algebraic and analytic aspects of the categories $\mathbf{Hilb}_{\mathbb{R}}$, $\mathbf{Hilb}_{\mathbb{C}}$ and $\mathbf{Hilb}_{\mathbb{H}}$ of real, complex and quaternionic Hilbert spaces and bounded linear maps. Additional examples include:

- for each von Neumann algebra A, the category $\mathbf{Hilb}_A$ of self-dual Hilbert A-modules; and,
- for each group G, the category $\mathbf{URep}_G$ of unitary representations of G.

Hilbert $*$-categories are "analytically" complete in two ways:

(i) every bounded increasing net of Hermitian endomorphisms has a supremum, and

(ii) every suitably bounded orthogonal family of parallel morphisms is summable.

These "analytic" completeness properties are not assumed outright; rather, they are derived, respectively, from two new universal constructions:

(i) codirected ℓ^2-limits of contractions, and

(ii) ℓ^2-products.

In turn, these universal constructions are built from directed colimits in the wide subcategory of isometries.

[1] M. Di Meglio and C. Heunen, *Hilbert $*$-categories: Where limits in analysis and category theory meet*, preprint arXiv:2505.17432, 2025.

[2] M. Di Meglio, *Pre-Hilbert $*$-categories: The Hilbert-space analogue of abelian categories*, preprint arXiv:2312.02883, 2025.

The Morita $(\infty, 2)$-Category of a Monoidal Category as a 2-Complicial Set

Arghan Dutta

Joint work with: Stefano Luneia, Martina Rovelli, Sam Silver

We present an explicit and elementary construction of the Morita $(\infty, 2)$-category associated to a monoidal category $\mathcal{C}^{\otimes}$ satisfying minimal conditions [1]. This construction provides a self-contained approach to understanding the higher categorical structure formed by monoids, bimodules, and bimodule maps.

Our main result constructs this structure as a 3-coskeletal 2-complicial set $M^{\natural}(\mathcal{C}^{\otimes})$—a model for $(\infty, 2)$-categories based on marked simplicial sets due to Verity [2]. In this construction, vertices encode monoids, edges represent bimodules, triangles capture bimodule maps $\varphi_{012} : M_{01} \otimes_{A_1} M_{12} \to M_{02}$ from balanced tensor products, and tetrahedra encode coherence conditions of the form

$$(\varphi_{012} \otimes_{A_2} M_{23}) \bullet \varphi_{023} = \alpha_{M_{01}|M_{12}|M_{23}} \bullet (M_{01} \otimes_{A_1} \varphi_{123}) \bullet \varphi_{013}.$$

The marking distinguishes invertible structure: marked edges correspond to invertible bimodules, while marked triangles represent bimodule isomorphisms.

The key technical requirement is that $\mathcal{C}^{\otimes}$ admits a *calculus of balanced tensor products*—conditions ensuring that coequalizers of the form

$$M \otimes_B N := \operatorname{coeq}\left[(M \otimes B) \otimes N \underset{\alpha_{M,B,N} \bullet (M \otimes \ell_N)}{\overset{r_M \otimes N}{\rightrightarrows}} M \otimes N \right]$$

exist and that functors $M \otimes (-)$ and $(-) \otimes P$ preserve them appropriately. This framework encompasses important examples including $\mathbf{Ab}^{\otimes}$, $\mathbf{Set}^{\amalg}$, and more generally, cocomplete closed monoidal categories.

Rather than directly verifying the bicategory axioms (a lengthy but well-believed result), we leverage the combinatorics of simplicial sets to reformulate these coherence conditions. We prove that $M^{\natural}(\mathcal{C}^{\otimes})$ satisfies the defining properties of a 2-complicial set by establishing the appropriate lifting properties against complicial horns $\Lambda^k[m] \hookrightarrow \Delta^k[m]$, thinness extensions $\Delta^k[m]' \hookrightarrow \Delta^k[m]''$, and saturation extensions $\Delta[3]^{\mathrm{eq}} \star \Delta[\ell] \hookrightarrow \Delta[3]^{\sharp} \star \Delta[\ell]$.

This approach offers both a proof of concept for efficiently encoding higher categorical structure and a pathway toward scaling these methods to even higher dimensions, such as when treating braided or symmetric monoidal categories. Our construction recovers known structures—such as $M^{\natural}(\mathbf{Ab}^{\otimes}) \simeq N_{\mathrm{Duskin}}(\mathbf{Alg}^{\mathbf{bi}})$ [3] and $M^{\natural}(\mathbf{Set}^{\amalg}) \simeq N_{\mathrm{Duskin}}(\mathbf{Span})$—while providing an explicit, elementary foundation that avoids heavy machinery.

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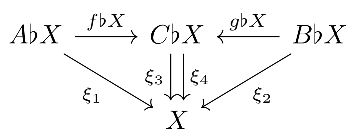
Uniqueness of actions in algebraically coherent categories

Arnaud Duvieusart

In the category of groups, an action of C on X is uniquely determined by a morphism $C \to \mathrm{Aut}(X)$. In particular, if A, B are subobjects of C such that $A \vee B = C$, then an action is determined by its restrictions to A and B.

The notion of internal action can be generalized to any semi-abelian category [1]; but in general this fact is no longer true. In [4], the authors introduced the *uniqueness of actions* condition for semi-abelian categories :

(UA) given a jointly strongly epimorphic cospan $A \xrightarrow{f} C \xleftarrow{g} B$, then for any actions $\xi_1, \xi_2, \xi_3, \xi_4$ on X, if the diagram



commutes, then $\xi_3 = \xi_4$.

Thus the category of groups, and more generally any *category of interest*, satisfies (UA).

We show that if a semi-abelian category $\mathcal{C}$ is *algebraically coherent* [3], then it satisfies a restricted version of (UA); namely, the condition above holds for jointly epimorphic cospans such that one leg has normal image. We also show that this restricted case is already sufficient to obtain several applications of the (UA) condition previously considered, such as those of [4, 2].

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[2] A. Cigoli, A. Duvieusart, M. Gran and S. Mantovani, *Galois theory and the categorical Peiffer commutator*, Homology, Homotopy and Applications **19** (2017), no. 1, 181—207

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Exponentiable Virtual Double Categories and Representability of Exponentials

Ea E T

Joint work with: Kevin Carlson

Double categories, which can be viewed as categories with two types of maps that interact through cells, have proved invaluable throughout both pure and applied mathematics for modelling structures with two relational modes. In many applications of interest, such as categories with maps given by functors and profunctors, it is necessary to allow one of the types of maps to only satisfy composition axioms up to coherent isomorphisms, resulting in the notion of pseudo-double categories. However, in these instances many important constructions on pseudo-double categories, such as the internal hom objects of Paré [1], fail to produce pseudo-double categories in general, and instead produce virtual double categories, where now one of the types of morphisms has a priori no composition operation. Nonetheless, virtual double categories have proved to still encode rich mathematical structures, providing an effective framework for formal category theory, from characterizations of adjoints and liftings to descriptions of pointwise Kan extensions and weighted (co)limits [2].

In studying virtual double categories themselves, the importance of functor categories in ordinary category theory motivates us to study when such functor category-type objects exist for virtual double categories. In this talk, we answer this question by providing a number of equivalent descriptions of the virtual double categories $\mathbb{D}$ for which the internal hom functor $(-)^{\mathbb{D}}$ exists, many of which address the prongs of a conjecture posed by Arkor in [3]. We will also show that one of our characterizations readily extends to a description of the virtual double functors which admit dependent products, generalizing the Conduché condition for exponentiable functors [4]. Throughout we will provide examples of exponentiable virtual double categories, including pseudo-double categories and cospan virtual double categories, along with connections to the theory of exponentiable multicategories, which can be seen as a shadow of the higher categorical theory of exponentiable virtual double categories. We will conclude by discussing extensions of Paré's work on the representability of hom objects for pseudo-double categories [1] to the case of virtual double categories, where the source of the hom object is exponentiable and the target is a weakly representable virtual double category. These extensions will be shown to provide a generalization of Day convolution and Day's classification of pro-monoidal categories (i.e. exponentiable multicategories) in terms of bi-cocontinuous monoidal structures on copresheaf categories [5].

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N-fold groupoids and n-groupoids in regular Mal'tsev categories

Nadja Egner

Joint work with: Marino Gran

Mal'tsev categories are a central concept in categorical algebra. In [2], a Mal'tsev category is defined as a finitely complete category $\mathbb{C}$ in which any internal reflexive relation is an internal equivalence relation. A regular category $\mathbb{C}$ is a Mal'tsev category if and only if the composition of internal equivalence relations on any object in $\mathbb{C}$ is commutative. Examples of Mal'tsev categories are given by any abelian category, the dual of any elementary topos, and any semi-abelian category such as the categories of groups and of Heyting semi-lattices.

Internal structures in a Mal'tsev category are well-behaved. Whereas it is in general not true that the category $\mathrm{Cat}(\mathbb{C})$ of internal categories in a (Barr-)exact category $\mathbb{C}$ is again exact, it is shown in [4] that $\mathrm{Cat}(\mathbb{C})$ is an exact Mal'tsev category whenever $\mathbb{C}$ is exact Mal'tsev. Moreover, any internal reflexive graph in a Mal'tsev category admits at most one internal category structure, which yields automatically an internal groupoid.

The category $\mathrm{Cat}^n(\mathbb{C})$ of internal n-fold categories in a finitely complete category $\mathbb{C}$ can be defined recursively as the category $\mathrm{Cat}(\mathrm{Cat}^{n-1}(\mathbb{C}))$ of internal categories in the category $\mathrm{Cat}^{n-1}(\mathbb{C})$ of internal $(n-1)$-fold categories in $\mathbb{C}$. In [1], also the category n-$\mathrm{Cat}(\mathbb{C})$ of internal n-categories in $\mathbb{C}$ is defined recursively. For $\mathbb{C} = \mathrm{Set}$ being the category of sets, this yields the usual definitions of n-fold category and n-category, respectively.

These recursive descriptions help us to generalize our results in [3] on the relation between $\mathrm{Cat}^2(\mathbb{C}) = \mathrm{Grpd}^2(\mathbb{C})$ and 2-$\mathrm{Cat}(\mathbb{C}) = 2$-$\mathrm{Grpd}(\mathbb{C})$, where $\mathbb{C}$ is a regular Mal'tsev category with finite colimits. We show that n-$\mathrm{Grpd}(\mathbb{C})$ is a Birkhoff subcategory, i.e. a reflective subcategory closed under subobjects and quotients, of $\mathrm{Grpd}^n(\mathbb{C})$. In consequence, n-$\mathrm{Grpd}(\mathbb{C})$ is regular Mal'tsev, exact Mal'tsev or semi-abelian whenever $\mathbb{C}$ is so. Using the results in [5], we can also show that n-$\mathrm{Grpd}(\mathbb{C})$ is action representable whenever $\mathbb{C}$ is semi-abelian, action representable, algebraically coherent and with normalizers.

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Comonads as Spaces

Aaron David Fairbanks

Joint work with: Kevin Carlson, David I. Spivak

Monads on the category **Set** provide a simple and unifying notion of algebra: groups, rings, and vector spaces are all examples of algebraic structures corresponding to monads. In the upcoming work [3] we explore the dual idea that comonads provide a simple and unifying notion of space.

First, topological spaces are identified with certain comonads on **Set**. We characterize these as precisely the density comonads of diagrams of subsets of a set (determining topological spaces as subbases). Thus the concept of topological space falls out of the theory of comonads, without mention of unions or finite intersections.

More general than topological spaces — but still not as general as arbitrary comonads on **Set** — are *toposes equipped with a set of enough points* [6, C2.2], or equivalently *ionads* as introduced by Garner [7]. Ionads at once generalize both topological spaces and small categories. An ionad amounts to a set X and a finite-limit-preserving comonad on $\mathbf{Set}^X$. They may also be identified with pullback-preserving comonads on **Set** itself. In particular, this recovers Ahman and Uustalu's result [2] that small categories are identified with certain comonads on **Set**, namely those with polynomial carrier.

Following Garner's insightful work, we generalize aspects of the theory of topological spaces to arbitrary comonads, not only on **Set**, but on arbitrary categories. We give categorical definitions of basis, subbasis, and continuous map. We also define a double category of comonads on a fixed category, in which the two types of arrows are such continuous maps and ordinary comonad morphisms. Restricting to those comonads on **Set** corresponding to categories recovers the double category of functors and retrofunctors of Clarke and Di Meglio [4].

In the field of coalgebra [9], coalgebras are used to model transition systems, dynamical systems, automata, and various infinite data types [5]. Whereas monads correspond to *varieties* of algebras specified by equations as in universal algebra, comonads correspond to *covarieties* of coalgebras specified by coequations [1]. A potential application of the theory we develop is to approach universal coalgebra using tools from topology. As a start in this direction, we are able to recover a foundational assumption in coalgebra, namely the preservation of weak pullbacks [8], from a topological perspective. More precisely, we characterize weak-pullback-preserving comonads on **Set** as the density comonads of diagrams with co-confluent category of elements, a generalization of the ordinary definition of basis from topology, and we show that all such diagrams are bases according to our more abstract definition.

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Linearity for monoidal structures

Roy Ferguson

Joint work with: Zurab Janelidze

This talk consists of two parts. In the first part, we explore partial linearity of categories in the following sense. We consider categories $\mathbb{C}$ equipped with a monoidal sum structure $(\oplus, 0)$ in the sense of [3] and the dual, a monoidal product structure $(\otimes, 1)$. In this setting, morphisms $X_1 \oplus \cdots \oplus X_n \to Y_1 \oplus \cdots \oplus Y_m$ are uniquely determined by matrices, as in linear categories in the sense of [4]. We further equip $\mathbb{C}$ with a natural transformation $i\colon X \oplus Y \to X \otimes Y$. Our first remark is that the mere existence of such i immediately forces pointedness of $\mathbb{C}$. We call i a *prelineariser* if it is compatible with the unitors of both monoidal structures in an appropriate sense. We show that i is a prelineariser if and only if the matrix corresponding to i is the identity matrix. We moreover show that a precursory coherence theorem holds: given any $\oplus$-word w and any $\otimes$-word v of the same length in the sense of [5], there exists a unique canonical morphism (with i included in the definition of canonical) from w to v, and that this morphism has matrix presentation the identity matrix. We call a category equipped with the above-mentioned structure, where i is a prelineariser, a *prelinear category*. When i is invertible, we call it a *lineariser* and we then speak of a *partially linear category*. We establish the full coherence theorem for partially linear categories.

When the sum structure is given by coproduct and the product structure is given by the Cartesian product, partially linear categories become linear categories. When both the sum structure and the product structure are given by the Cartesian product, partially linear categories become weakly unital categories [6], which include all unital categories [1]. In the second part of the talk we extend results on centrality in unital categories to our general context. Extending the concept of a central morphism introduced in [1], we say that a morphism $f\colon X \to Y$ in a prelinear category is *central* if the morphism $\left[\begin{smallmatrix} 1 & f \\ 0 & 1 \end{smallmatrix}\right] : X \oplus Y \to X \otimes Y$ exists. We show that in the partially linear context central morphisms admit "enrichment" in **Mon** as in the case of unital categories. A preprint detailing the results of this talk can be found at [2].

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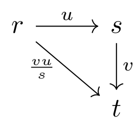
Topos polar decomposition

Jon Funk

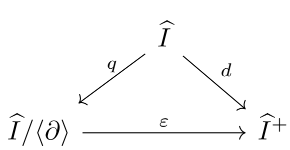
Background and motivation: a polar decomposition of an element T of a C^*-algebra is a product $T = UA$ where U is a partial isometry, A is positive, and $\mathrm{Ann}(U) = \mathrm{Ann}(A)$, where $\mathrm{Ann}(U) = \{ X \mid UX = 0 \}$. Such a decomposition is necessarily unique and we have $A = |T| = \sqrt{T^*T}$. Let us say that a C^*-algebra admits *polar decomposition in the traditional sense* if every element of it has such a polar decomposition. If a C^*-algebra admits polar decomposition in this sense, then it has the following property: for every element T there is a projection P such that $\mathrm{Ann}(T) = \mathrm{Ann}(P)$ (the algebra is said to be Rickart). However, not every C^*-algebra is Rickart, so we would like to broaden the scope of polar decomposition by seeking a generalization of it that agrees with the traditional notion in the Rickart case. It turns out that the criteria we find, which we shall call *topos polar decomposition*, unfolds as a representation of the (multiplicative) cosets of the partial isometries of the algebra by the positive elements.

A topos approach: the objects of the division category $\mathcal{M}$ associated with a unital ring (with unit 1) are the elements of the ring, denoted r, s, etc. A morphism $u : r \to s$ of $\mathcal{M}$ is an element u of the ring such that $\mathrm{Ann}(u) = \mathrm{Ann}(r)$, and $\exists a\, u = sa$, which we write $s \mid u$. For instance, if $s \mid r$, then $r : r \to s$ is a morphism of $\mathcal{M}$. Composition in $\mathcal{M}$ is as follows, which makes sense because $s \mid u$: if $u = sa$, then $\frac{vu}{s} = va$.

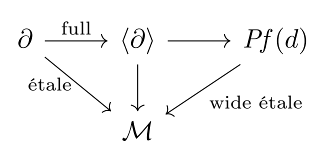


Let us say that a subcategory $\mathcal{C} \to \mathcal{M}$ is *étale* if: i) for any two objects r, s of $\mathcal{C}$, if $s \mid r$, then the morphism $r : r \to s$ is in $\mathcal{C}$, and ii) in a triangle of composable morphisms of $\mathcal{M}$ (as above) if $v, \frac{vu}{s} \in \mathcal{C}$, then $u \in \mathcal{C}$. A subcategory is *wide* if it has the same set of objects as the including category. Our analysis of polar decomposition uses the following basic tool: the wide étale subcategories of $\mathcal{M}$ correspond to quotients of the representable presheaf $\widehat{1}$ in the topos of presheaves on $\mathcal{M}$, where $\widehat{1}(r) = \mathcal{M}(r, 1) = \{ u \mid \mathrm{Ann}(u) = \mathrm{Ann}(r) \}$. The correspondence associates with a wide étale subcategory $\mathcal{C} \to \mathcal{M}$ its presheaf quotient of multiplicative cosets $\widehat{1} \to \widehat{1}/\mathcal{C}$. On the other hand, it associates with an element $x : \widehat{1} \to \mathcal{F}$ of a presheaf $\mathcal{F}$ on $\mathcal{M}$ what we call its principal fiber, denoted $Pf(x) \to \mathcal{M}$. By definition, the objects of $Pf(x)$ are the elements of the ring, with morphisms $u : r \to s$ such that $x \cdot r = x \cdot u$. The subcategory $Pf(x) \to \mathcal{M}$ is wide étale.

C^*-algebras: the partial isometries of a unital C^*-algebra (with unit I) collectively form a subcategory $\partial \to \mathcal{M}$ of the division category of the underlying ring of the algebra on the projections. The subcategory ∂ is étale, but it is not wide. Nevertheless, it generates a wide étale subcategory $\langle \partial \rangle \to \mathcal{M}$ depicted below (right). On the other hand, the positive elements of the algebra organize themselves as a quotient of the representable presheaf $\widehat{I}$, labeled d in the triangle below (left). It follows that the principal fiber $Pf(d)$ of d consists of all morphisms $U : R \to S$ of $\mathcal{M}$ such that $R^*R = U^*U$. Furthermore, ∂ is a full subcategory of $Pf(d)$, so that $\langle \partial \rangle$ is a subcategory of $Pf(d)$, and ∂ is full in $\langle \partial \rangle$.



$$q_R(U) = \langle \partial \rangle U \; ; \; d_R(U) = U^*U$$



The cosets of $\langle \partial \rangle$ form a quotient q depicted above, and moreover, there is a canonical natural transformation ε comparing q and d. Let us say that a C^*-algebra admits *topos polar decomposition* if ε is an isomorphism. This condition holds if and only if $\langle \partial \rangle = Pf(d)$.

Proposition: A C^*-algebra admits polar decomposition in the traditional sense if and only the algebra is Rickart, and it admits topos polar decomposition.

On the 2-category of symmetric 2-rigs

Nicola Gambino

Joint work with: Mathieu Anel, Marcelo Fiore, Richard Garner, Christina Vasilakopoulou

The notion of a 2-rig provides one possible way to categorify the notion of a rig, i.e. a ring without negatives. Here, colimits play the role of sums, tensor products play the role of products, and preservation of colimits by the tensor product corresponds to the distributive law of rings. Explicitly, a 2-rig is a monoidally cocomplete category, i.e. a cocomplete category equipped with a monoidal structure such that the tensor product preserves colimits in each variable. Symmetric 2-rigs, where the monoidal structure is symmetric, categorify commutative rigs.

The aim of the talk is to discuss some steps in the program of developing the theory of symmetric 2-rigs in analogy with commutative ring theory, as suggested and investigated by André Joyal. The motivation for this comes from several areas, including algebraic topology (via operads), geometry (via categories of quasi-coherent sheaves), mathematical logic and theoretical computer science (via analytic functors). Indeed, examples of symmetric 2-rigs abound. In particular, we have free 2-rigs (obtained by taking presheaves on free symmetric monoidal categories), semi-free 2-rigs (obtained by taking presheaves over symmetric monoidal categories), operadic 2-rigs (obtained by taking presheaves over symmetric monoidal categories associated to operads), and 'symmetric algebra' 2-rigs (obtained by freely adding a symmetric tensor product to a locally presentable category).

First, I will describe some properties of the 2-category of symmetric 2-rigs, which are quite analogous to those of the category or commutative rings, although unavoidably more subtle, as they involve 2-dimensional monad theory rather than standard monad theory. I will also illustrate how the operation of 'taking points' of a symmetric 2-rig gives rise to a duality.

Secondly, I will consider two full sub-2-categories of 2-category of symmetric 2-rigs and identify them with the bicategory of categorical symmetric sequences [2] and the bicategory of operads and bimodules [4]. The latter is of particular interest since, as will be explained, the Boardman-Vogt tensor product of operads can be shown to act also on bimodules [5, 6], extending work of Dwyer and Hess [1].

Finally, I will outline how the analogy with commutative ring theory suggests the idea of exploring counterparts of the notions of a derivation and of the module of Kähler differentials, showing how the work in [3] fits in this framework and can be extended further.

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Toposes with enough points as categories of étale spaces

Sam van Gool

Joint work with: Jérémie Marquès, Umberto Tarantino

Barr [1] showed that topological spaces correspond to relational modules for the ultrafilter monad β on Set. The aim of this talk is to discuss a recent lifting of this result to Grothendieck toposes with enough points. More precisely, when $\mathcal{E}$ is such a topos, we equip the category $\mathsf{pt}\,\mathcal{E}$ of its points with a relational generalization of Makkai's ultrastructure [8]. We propose to call the resulting new concept of a 'category equipped with relational ultrastructure' an *ultraconvergence space*. We argue that ultraconvergence spaces are an appropriate categorical generalization of Barr's relational modules for β. Once the topological notions of continuous and étale maps are also naturally extended to this setting, our main theorem reads as follows:

Theorem. Let $\mathcal{E}$ be a Grothendieck topos with enough points. Then

$$\mathcal{E} \simeq \mathbf{Cont}(\mathsf{pt}\,\mathcal{E}, \mathsf{Set}) \simeq \mathbf{Etale}(\mathsf{pt}\,\mathcal{E})\ .$$

That is, $\mathcal{E}$ is equivalent to the topos of continuous maps from $\mathsf{pt}\,\mathcal{E}$ to the ultraconvergence space Set, and to the topos of étale ultraconvergence spaces over $\mathsf{pt}\,\mathcal{E}$.

Our work extends Makkai's duality [8] between coherent toposes and ultracategories, and our proof simultaneously generalizes and simplifies Makkai's original proof. In logical terms, our theorem is a strong conceptual completeness theorem for geometric theories with enough models in Set.

This talk is based on our recent preprint [5]. The same result has recently been obtained independently by both G. Saadia [9] and A. Hamad [6], albeit via a rather different route. Notably, the proofs in [9, 6] both make crucial use of a classical representation theorem for toposes via topological groupoids [7, 2]. Our proof of the theorem, on the other hand, is obtained without relying on any such result. Time permitting, we will also touch on connections to ionad theory [3, 4], and describe some future directions to explore.

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Hopf formulae for cocommutative Hopf algebras

Marino Gran

Joint work with: Andrea Sciandra

In recent years, many new applications of categorical Galois theory have emerged in various interesting non-abelian algebraic contexts. In particular, in the realm of semi-abelian categories [1], this approach has led to some new calculations of higher fundamental groups in terms of generalized commutators in categories such as that of compact groups, crossed modules, and skew braces [2], among many others. These categories share many structural properties with the categories of groups and of Lie algebras. The category of cocommutative Hopf algebras over a field is also semi-abelian [3], raising the natural question of whether some similar homological methods can be applied to study such structures as well.

In this talk, after reviewing some fundamental properties of semi-abelian categories and some motivating examples, I'll then explain that the answer to the above question is affirmative. Indeed, the exactness properties of cocommutative Hopf algebras together with the existence of the celebrated Takeuchi's free functor described in [4] - universally associating a Hopf algebra with any coalgebra - make it possible to establish some new Hopf-type formulae for the homology of cocommutative Hopf algebras [5]. Note that in this work an important role is actually played by the *cleft extensions*, namely by those surjective morphisms of Hopf algebras that are split as coalgebra morphisms. Cleft extensions satisfy all the needed properties in order to build a weak universal central extension of any given cocommutative Hopf algebra. Moreover, with any cleft extension, one can associate a 5-term exact sequence in homology that can be seen as a Hopf-theoretic analogue of the classical Stallings-Stammbach exact sequence in group theory.

This new approach can also be applied to investigate the homology of cocommutative Hopf braces [6], which are interesting structures that naturally occur in the study of the solutions of the so-called quantum Yang-Baxter equation. The category of cocommutative Hopf braces turns out to be both semi-abelian and strongly protomodular [7]. It is also monadic on the category of coalgebras [8], so that it is possible to investigate it from the perspective of non-abelian homological algebra.

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Semisimple Hopf monoidal categories are group theoretical

David Green

Joint work with: Brett Hungar, Sean Sanford

The celebrated Tannaka-Krein reconstruction theorem [Del07] provides an equivalence between well-behaved monoidal categories equipped with a functor to the category of vector spaces, and the category of bialgebras. This equivalence, itself being a monoidal functor, can be used to transport various structures and properties between the two contexts.

Motivated by both physical and purely mathematical considerations, the Tannaka-Krein reconstruction theorem was expected to categorify as early as 1997 (see [Neu97]) and proven to do so for multifusion Hopf monoidal categories in[Gre23]. Here, multifusion Hopf monoidal categories are the categorical analogue of the finite dimensional semisimple Hopf algebras.

The surprise is that Hopf monoidal structures on multifusion categories admit a simple classification (categorifying the results of Natale [Nat03]) in terms of exact factorizations of finite groups and group cohomology, in marked contrast to the decategorified situation.

This classification is obtained as a synthesis of results of Thibault Deccopet and Matthew Yu [DY25], and the previous work [Gre23]. Moreover, a connection with the Kac exact sequence [Kac68] provides a reasonable ansatz about the classification of semisimple Hopf monoidal n-categories.

In the talk, we will introduce a definition of Hopf monoidal categories purely in terms of monoidal 1-categories, and provide some motivation for these categorical structures as well as a statement of the classification result.

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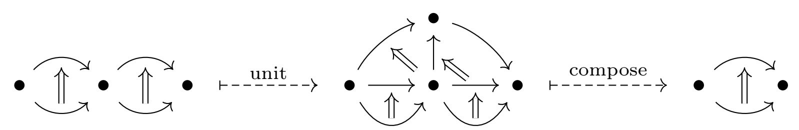
Semi-strictification of (∞, n)-categories

Amar Hadzihasanovic

Joint work with: Clémence Chanavat

In this talk, I will present my joint work with Clémence Chanavat [1] providing the first (to our knowledge) equivalence between a fully weak, non-algebraic model and a semi-strict algebraic model of (∞, n)-categories. Since the models satisfy the homotopy hypothesis in the case $n = 0$, this also exhibits the first semi-strict model of homotopy types with algebraic units and composition.

Since the semi-strict model satisfies a strict form of associativity and interchange, while unitality laws only hold up to coherent equivalence, this is a semi-strictification result in the spirit of Carlos Simpson's "weak units" conjecture [2], although it does not map neatly to a classification in terms of globular algebra: the model separates the formation of *pasting diagrams* (which uses globular pasting operations) from *composition*, which is restricted to the subclass of *round diagrams*, similar to Simon Henry's notion of composition for *regular* ω-categories [3]. Globular composition operations are derived from a combination of "padding up pasting diagrams with units" and round diagram composition, as in the following picture:



Both the definition of the semi-strict model and the semi-strictification result are founded on a "category of diagram shapes" that plays the same role that Joyal's Θ category plays in the theory of strict ω-categories, but in which—unlike in Θ—morphisms that classify *face*, *unit*, and *composition* operations are neatly separated into a ternary factorisation system. This category is based on the combinatorial theory of regular directed complexes [4] and a further study of classes of morphisms between regular directed complexes. Suitably limit-preserving presheaves on the restriction to (face, unit)-morphisms support weak (partially non-algebraic) models of (∞, n)-categories, and freely extending with composition-morphisms realises semi-strictification, following a proof strategy first sketched in [5]. More technically, semi-strictification is exhibited by acyclic cofibrations constituting the derived unit components of a Quillen equivalence between weak model categories [6] whose fibrant objects are, respectively, the weak (∞, n)-categories and (up to an acyclic fibration) the semi-strict ones; weak functors lift to functors which strictly preserve composition, but only weakly preserve units. The constructions are explicit and combinatorial, in the spirit of Mac Lane's strictification of bicategories.

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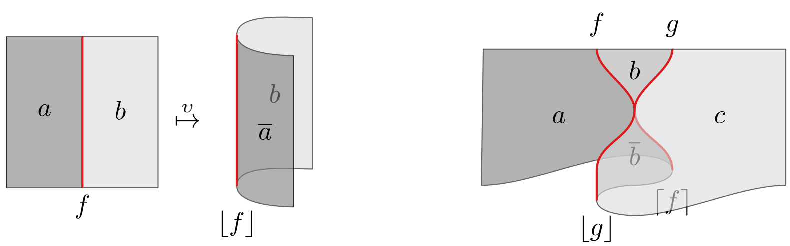
Nuclearity and Trace in Monoidal Bicategories and Extended Conformal Field Theories

James Hefford

Nuclear and trace ideals provide a way of formalising, respectively, partial compact closure and partial traces in monoidal categories [1]. Hilbert spaces and bounded linear maps provide the canonical example of a category with a nuclear and trace ideal: the Hilbert-Schmidt maps and the trace class maps respectively.

In this talk, I will give a more modern formulation of nuclear and trace ideals in terms of profunctors and use this to categorify the notions to the setting of monoidal bicategories $\mathcal{B}$ making use of the theory of pseudocoends [4]. A trace 2-ideal is a monoidal sub-biprofunctor of the hom $T(-,-) \hookrightarrow \mathcal{B}(-,-) : \mathcal{B} \nrightarrow \mathcal{B}$ together with a functor $\mathrm{tr} : \oint^a T(a,a) \to \mathcal{B}(i,i)$, and a certain higher coherence cell ensuring the trace is monoidally well-behaved. Trace 2-ideals capture the part of a monoidal bicategory that permits a higher trace in the sense of the shadows of [5], and in the case that $T(-,-) = \mathcal{B}(-,-)$, I will show that the coherences of the pseudocoend coincide with those of [5].

A nuclear 2-ideal for $\mathcal{B}$ consists of a monoidal sub-biprofunctor of the hom $N(-,-) \hookrightarrow \mathcal{B}(-,-)$ equipped with a monoidal pseudonatural isomorphism, $\upsilon : N(a^\dagger, b) \cong \mathcal{B}(i, \overline{a} \otimes b)$ and invertible 2-cells witnessing the yanking equation, together with coherences including the swallowtail equation familiar from the theory of compact closed bicategories [2]. Indeed, when $N(-,-) = \mathcal{B}(-,-)$ then $\mathcal{B}$ is a compact closed bicategory.



As the main example, I will show that the bicategory 2Hilb of infinite dimensional 2-Hilbert spaces has a nuclear and a trace ideal categorifying the story for Hilb. Similarly, the bicategory Bord of conformal cobordisms permits a nuclear and a trace ideal and I will use this to give a formalisation of a once extended conformal field theory as a nuclear 2-functor, $Z : \mathsf{Bord} \to \mathsf{2Hilb}$. This categorifies the result of [3] and formalises some results from [6].

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Models of set-theory from elementary 2-topoi

Joseph Helfer

There is a long tradition of comparing elementary topos theory with traditional $\in$-based set theories such as ZF, beginning with [1, 5, 6]. These two kinds of set theories are well-known to be "mismatched", in that ZF allows unbounded quantification over sets, whereas this is not possible in the internal logic of a topos. The result is that, from an elementary topos, one can construct a model of only a weak fragment of ZF, in which only bounded quantification is allowed.

Two approaches that overcome this limitation are the "Algebraic Set Theory" of Joyal and Moerdijk [4] and the more recent "Stack Semantics" of Shulman [7]. I will explain a different approach, related to these two, to obtain models of set theory "topos-theoretically", based on the notion of *elementary 2-topos* introduced in [8], and which I have been further developing in [2, 3] following ideas of M. Makkai and B. Boshuk.

The notion of 2-topos is an axiomatization of the basic elementary properties of the 2-category of categories, much as the notion of 1-topos is related to the category of sets. The central axiom stipulates that a 2-topos has a special object $\mathcal{S}$, playing the role of the 1-category of sets. Thus, 2-topos theory is akin to class-set theory, the objects of the 2-topos being the "classes" (or rather, "large categories"), and the objects of $\mathcal{S}$ being the "sets". As I will explain, from the rather simple and natural axioms of an elementary 2-topos, one can produce a model of full (intuitionistic) ZF set theory. This suggests 2-topoi as a natural environment for the algebraic study of set-theoretic universes.

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Monoidal Differential Turing Categories

Isaiah B. Hilsenrath

Joint work with: Jean-Simon Pacaud Lemay

Differential λ-calculus [6] is an extension of λ-calculus which enables one to treat differentiation of analytic functions purely syntactically. Just as simply typed λ-calculus is sound and complete with respect to Cartesian closed categories by the Curry-Howard-Lambek correspondence, simply typed differential λ-calculus is sound and complete with respect to Cartesian (closed) differential categories [3]. In [4], Cockett and Gallagher refined these ideas to provide a categorical semantics of the *untyped* differential λ-calculus, introducing the Cartesian differential Turing category with differential canonical codes. This coherently combined a Cartesian differential category with a type of category that provides a sound and complete semantics for the ordinary untyped λ-calculus: a (total) Turing category with canonical codes.

Now, an important source of examples of Cartesian differential categories comes from the categorical semantics of differential linear logic: monoidal differential categories [2]. Briefly, a monoidal differential category is a symmetric monoidal category with a comonad ! and a natural isomorphism $d_A \colon\, !A \otimes A \to\, !A$, called the deriving transformation, that satisfies certain coherences that capture the fundamental properties of differentiation, like the product rule and chain rule. The coKleisli category of a monoidal differential category is a Cartesian differential category [1]. It is then natural to ask what is the analogue of a monoidal differential category whose coKleisli is a Cartesian differential Turing category.

In this talk, I will introduce monoidal differential Turing categories and their analogous notion of differential canonical codes. At the heart of this definition is a distinguished object T, called the universal object, a family of application morphisms $\bullet \colon T \otimes\, !B \to C$, and a family of functions $\lambda\colon \operatorname{Hom}(A \otimes\, !B, C) \to \operatorname{Hom}(A, T)$ such that the following diagrams commute:

$$\begin{array}{ccc} T \otimes\, !B & \xrightarrow{\ \bullet\ } & C \\ {\scriptstyle \lambda(f)\otimes \mathrm{id}_{!B}} \uparrow & \nearrow_{f} & \\ A \otimes\, !B & & \end{array} \qquad \begin{array}{ccc} & & T \\ & {\scriptstyle \lambda((g\otimes \mathrm{id}_{!B});f)} \nearrow & \uparrow {\scriptstyle \lambda(f)} \\ D & \xrightarrow{\ g\ } & A. \end{array}$$

One should think of these diagrams as defining a weak form of currying: λ is transposition, $\bullet$ is evaluation which is also linear in its first argument, and T behaves like a uniform exponential object. We will show that the coKleisli category of a monoidal differential Turing category (with differential canonical codes) is indeed a Cartesian differential Turing category (with differential canonical codes). We will then extend these constructions to the closely-related *reverse* differential setting from [5].

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Exponentiable morphisms for a clan

Joseph Hua

Joint work with: Reid Barton

Many categories, such as Top and Cat, are not locally Cartesian closed, but still have certain exponentiable morphisms [1, 2]. We propose a more fine-grained condition of being exponentiable relative to a *clan* [5].

A *clan* $(\mathcal{C}, \mathcal{R})$ consists of a category $\mathcal{C}$ with a class of maps $\mathcal{R}$ whose members are called $\mathcal{R}$-*maps*, such that pullbacks of $\mathcal{R}$-maps along all morphisms exist and are $\mathcal{R}$-maps, all isomorphisms are in $\mathcal{R}$, and $\mathcal{R}$ is closed under composition. Modifying the original definition in [5], we do not require that $\mathcal{C}$ has a terminal object, nor that maps to the terminal object are $\mathcal{R}$-maps. Following [5], we use $\mathcal{R}(X)$ to denote the full subcategory of the slice $\mathcal{C}/X$ consisting of objects whose underlying map is in $\mathcal{R}$.

Fix a clan $(\mathcal{C}, \mathcal{R})$ and $f : Y \to X$ a morphism in $\mathcal{C}$ with all pullbacks, and such that for any pullback $f' : Y' \to X'$ of f, the restricted pullback functor $(f')^* : \mathcal{R}(X') \to \mathcal{R}(Y')$ has a right adjoint $f'_* : \mathcal{R}(Y') \to \mathcal{R}(X')$. We say the *Beck–Chevalley condition holds at f* if every pullback of f induces a canonical isomorphism:

$$\begin{array}{ccc} Y' & \xrightarrow{s'} & Y \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{s} & X \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \mathcal{R}(Y') & \xleftarrow{(s')^*} & \mathcal{R}(Y) \\ {\scriptstyle f'_*}\downarrow & \cong & \downarrow{\scriptstyle f_*} \\ \mathcal{R}(X') & \xleftarrow{s^*} & \mathcal{R}(X) \end{array}$$

We say the *partial right adjoint condition* holds at $f : Y \to X$ when there is a bijection of hom-sets

$$\mathrm{Hom}_{\mathcal{C}/Y}(f^* X', A) \cong \mathrm{Hom}_{\mathcal{C}/X}(X', f_* A)$$

natural in $X' \in (\mathcal{C}/X)^{\mathsf{op}}$ and $A \in \mathcal{R}(Y)$. Note that this is stronger than f_* merely being right adjoint to the restricted functor $f^* : \mathcal{R}(X) \to \mathcal{R}(Y)$.

Theorem 1. *The following are equivalent. (i) The Beck–Chevalley condition holds at all pullbacks of f. (ii) The partial right adjoint condition holds at all pullbacks of f. (iii) The partial right adjoint condition holds at f and the pushforward functor (in presheaves) $\Pi_f : \widehat{\mathcal{C}}/Y \to \widehat{\mathcal{C}}/X$ preserves (Yoneda images of) $\mathcal{R}$-maps.*

The map $f : Y \to X$ is called $\mathcal{R}$-*exponentiable* if it satisfies the equivalent conditions in Theorem 1. Note that f itself need not belong to $\mathcal{R}$. When $\mathcal{C}$ has finite limits, this notion is a special case of properness with respect to a fibration over $\mathcal{C}$ [6], namely the fibration corresponding to the indexed category $\mathcal{R}(-)$. A prototypical example is $\mathcal{C} = \mathrm{Top}$ and $\mathcal{R}$ the class of local homeomorphisms, so that $\mathcal{R}(X) \simeq \mathrm{Sh}(X)$. Then the proper base change theorem [3] says that proper maps are $\mathcal{R}$-exponentiable. An example of similar flavor: in $\mathcal{C} = \mathrm{Cat}$, discrete fibrations are exponentiable with respect to the clan structure of discrete opfibrations (and vice versa). A different flavor of example is given by the category of simplicial sets with $\mathcal{R}$ the class of Kan fibrations: the $\mathcal{R}$-exponentiable maps are exactly the *sharp maps* of Rezk [4].

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A formal category theoretic approach to the homotopy theory of dg categories

Yuki Imamura

A *differential graded category* (or *dg category*) is an enriched category over the symmetric monoidal closed category of cochain complexes of modules (over a field, for simplicity). It is widely used in algebraic geometry and representation theory as an enhancement of triangulated categories ([1, 6]), providing a richer underlying structure. The homotopy theory of complexes up to quasi-isomorphism induces a natural homotopical structure on dg categories ([5]), whose weak equivalences are called *quasi-equivalences*. Accordingly, one can construct the localization $\mathsf{Ho}(\mathsf{dgCat})$ of the category of dg categories with respect to quasi-equivalences.

In this talk, we will present an approach to the homotopy theory of dg categories from the perspective of formal category theory. We introduce a bicategory that serves as a natural refinement of $\mathsf{Ho}(\mathsf{dgCat})$, and show that this bicategory carries the structure of a *proarrow equipment* in the sense of Richard J. Wood [3, 4]. Proarrow equipments provide a general framework for formal category theory and allow one to define notions of (co)limits in an abstract setting. Applying this framework to our proarrow equipment, we derive a notion of homotopical (co)limits in dg categories that is respected by quasi-equivalences. We show that these homotopical limits include homotopical shifts and cones, yielding a formal characterization of pretriangulated dg categories. As an application, we also establish reflection results concerning adjoints and colimits.

This talk is based on the paper [2].

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Distilling Monads

Brenda Johnson

Joint work with: Kristine Bauer, Kathryn Hess, and Julie Rasmusen

Functor calculi have become important tools for understanding structures in algebraic and geometric topology. A functor calculus provides a means of approximating functors by towers of "polynomial" functors, analogous to using Taylor polynomials to approximate functions of real numbers. A number of different types of functor calculi have been developed to address a broad range of questions, but each has been created on an ad-hoc basis. This work is part of a larger program to identify general categorical conditions and processes from which one can build new functor calculi.

As a starting point, we are focused on the methods used to construct the calculus towers of the discrete and abelian functor calculi of [1] and [2], and a dual process used to create the dual calculus cotower of [3]. These methods use comonads that act on a category of functors to construct degree n approximations in the case of the abelian and discrete calculi, and monads to create codegree n approximations in the case of the dual calculus. These monads and comonads are constructed via particular tools – homotopy colimits and limits – that are widely used in algebraic topology for their nice homotopy-theoretic properties. However, these homotopy-theoretic properties are not essential to the (co)monad-building process, suggesting that a much broader range of monads and comonads for building new functor calculi towers can be created by appropriately generalizing these techniques.

In this talk, we will present a generalization of homotopy colimits, which we call a *distillation system*, that is defined as a type of oplax transformation between two $\mathrm{Cat}^{\mathrm{op}}$-actegory structures on CAT. We will explain how these distillation systems encode some of the essential properties of homotopy colimits alluded to above, and the role these properties play in a general functorial process for using a distillation system to transform a strict monoidal functor $\theta : I \to \mathcal{B}$ and a $\mathcal{B}$-actegory $\mathcal{C}$ into a monad that acts on $\mathcal{C}$. This process recovers the monads used in the dual calculus, and offers new examples to explore. Future work will focus on using this process to produce sequences of monads to create new types of dual functor calculi.

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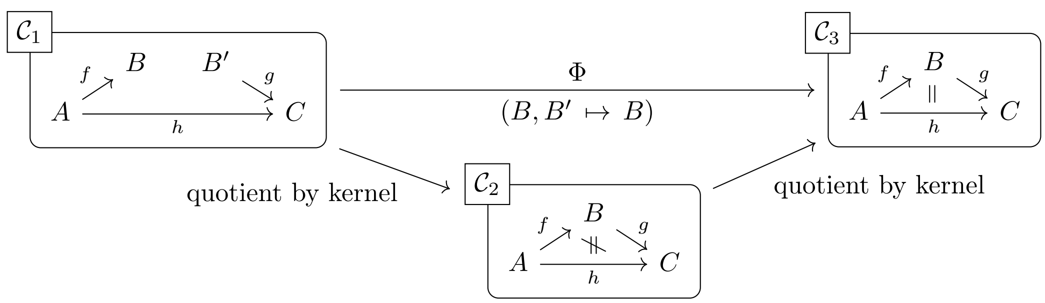
On the decomposition of a strong epimorphism into regular epimorphisms

Yuto Kawase

Joint work with: Hayato Nasu

The fundamental theorem of homomorphisms plays a central role in abstract algebra. It states that for every homomorphism, the quotient algebra modulo its kernel is isomorphic to its image, a subalgebra of its codomain. In category theory, the theorem can be reformulated as a decomposition of a morphism into a regular epimorphism (= quotient) followed by a monomorphism (= subalgebra). However, as the notion of algebras is generalized beyond classical equational ones, such a decomposition may not be possible. Indeed, the decomposition tends to fail in several categories of *partial algebras*, algebras including partially defined operations.

The category **Cat** of (small) categories is one of the typical examples of the category of partial algebras. Consider the following morphism (= functor) in **Cat**:



Then, the kernel of the functor Φ identifies the objects B and B' in $\mathcal{C}_1$, but there is no way to identify $g \circ f$ and h inside the domain category $\mathcal{C}_1$ because $g \circ f$ does not even make sense there. In short, the morphism collapses beyond what the domain category can see. Once B and B' get identified (in $\mathcal{C}_2$), $g \circ f$ comes to make sense, and taking another quotient by the equality $g \circ f = h$ gives rise to its image.

The *decomposition number* $\delta(f)$ for a morphism f, introduced in [1], is the number of iterations of taking quotients needed to obtain an object isomorphic to the image. For example, we have $\delta(\Phi) = 2$ in the above example. In fact, for an arbitrary functor Ψ, we have $\delta(\Psi) \leq 2$. Introducing a syntactic way to give an upper bound for the decomposition numbers, we will demonstrate several examples of calculating the supremum length of such decompositions and will present further generalizations. This talk is based on a paper [2] in preparation.

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Polynomial functors from Lawvere theories

Minkyu Kim

For a category $\mathcal{C}$, a $\mathcal{C}$-module is a functor from $\mathcal{C}$ to the category of k-modules. Here, k denotes a unital commutative ring unless otherwise specified. The *(polynomial) degree* of a $\mathcal{C}$-module serves as an invariant that allows for a systematic study of $\mathcal{C}$-modules when $\mathcal{C}$ is given as a monoidal category with zero unit. A polynomial $\mathcal{C}$-module is a $\mathcal{C}$-module with a finite polynomial degree. The origin of these notions goes back to Eilenberg and Mac Lane [1] where $\mathcal{C}$ is taken to be an additive category.

Several adjunctions between functor categories, respecting the degree of functors, have been studied.

1. Let R be a unital ring, and let $\mathbf{P}_R$ denote the category of finitely generated projective R-modules. There is a classical adjunction between the category of $\mathbf{P}_{\mathbb{Z}}$-modules and that of $\mathfrak{S}$-modules, where $\mathfrak{S}$ denotes the category of finite sets and bijections. When restricted to *analytic* $\mathbf{P}_{\mathbb{Z}}$-modules, and if $\mathrm{ch}(k) = 0$, then this adjunction induces an equivalence compatible with degree of $\mathbf{P}_{\mathbb{Z}}$-modules.
2. This example extends the previous one to general R. For $k = \mathbb{Z}$, Pirashvili [2] gave an equivalence between the category of $\mathbf{P}_R$-modules of polynomial degree at most d, modulo those of degree at most $d-1$, and the category of right $R \sim \mathfrak{S}_d$-modules. Here, $R \sim \mathfrak{S}_d$ denotes the wreath product.
3. In [3], Powell constructs an adjunction between the category of $\mathbf{F}^{\circ}$-modules and that of modules from the PROP associated with the Lie operad. Here, $\mathbf{F}^{\circ}$ denotes the opposite category of finitely generated free groups. Powell also established that, when restricted to *analytic* $\mathbf{F}^{\circ}$-modules, and if $\mathrm{ch}(k) = 0$, then the adjunction induces an equivalence, compatible with degree of $\mathbf{F}^{\circ}$-modules.

We develop a systematic method for constructing adjunctions that incorporate Powell's adjunction and the adjunction implicit in Pirashvili's equivalence. In addition to recovering these known examples, the present work also yields new adjunctions involving functor categories over Lawvere theories, compatible with polynomial degree. Recall that a Lawvere theory is a category with finite products, whose objects are $\mathbb{N}$, with the products on objects given by addition.

Main Result A.([5]) Given a Lawvere theory $\mathcal{C}$ with a zero object, we construct a natural k-linear PROP $\tilde{\Phi}_{\mathcal{C}}$, and establish an (explicit) adjunction between $\tilde{\Phi}_{\mathcal{C}}$-modules and $\mathcal{C}$-modules. Furthermore, if $\mathcal{C}$ satisfies some mild conditions, then (A) $\tilde{\Phi}_{\mathcal{C}}(n, m) = 0$ if $n < m$, and (B) the adjunction corresponds polynomial $\mathcal{C}$-modules to truncated $\tilde{\Phi}_{\mathcal{C}}$-modules, and conversely.

Main Result B.([5]) The result **A** recovers and refines known results by the following statements: (i) When $\mathcal{C}$ is the category of free R-modules of finite rank, $\tilde{\Phi}_{\mathcal{C}}$ can be described as a category built from wreath products. (ii) When $\mathcal{C}$ is the opposite category of finitely generated free nilpotent groups of class $\leq c$, $\tilde{\Phi}_{\mathcal{C}}$ is isomorphic to the PROP associated with the operad for nilpotent Lie algebras of class $\leq c$.

Remark. This work starts from a framework that the author presented at CT2024. The framework has also been applied to polynomial functor theory in a different context [4]. The long term goal of this line was to find an adjunction for functors on Habiro-Massuyeau category, which is currently under preparation.

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Double Categories for Operator Algebras and AQFT

Khyathi Komalan

Algebraic quantum field theory (AQFT), introduced by Rudolf Haag and Daniel Kastler, provides a framework for assigning an algebra of observables to regions of spacetime [1]. Categorically, an AQFT may be described as a covariant functor from a category of spacetime regions — typically open, bounded regions of Minkowski spacetime with inclusions as morphisms — into a category of operator algebras [2],

$$\mathcal{A} : \text{Regions} \longrightarrow \text{Corresponding Operator Algebras},$$

satisfying certain properties known as the Haag-Kastler axioms.

Fixing a region, one restricts to the assignment $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$, i.e. an AQFT net. In this way, the 1-functorial formulation makes precise how geometric inclusions and operator-algebraic structure interact.

However, thinking of an AQFT as a functor hides a second kind of movement. Namely, what happens when — rather than enlarging a region by an inclusion $U \subseteq V$ — we want to move the same region to a new place, by identifying U with a geometrically equivalent copy sitting inside V via an embedding $h\colon U \hookrightarrow V$? For this, we turn to double categories.

To construct an AQFT as a double functor, we introduce a double category of regions in Minkowski spacetime, with inclusions as vertical maps and admissible embeddings as horizontal maps, and take as codomain Juan Orendain's globularly generated double category of von Neumann algebras (a special class of operator algebras on Hilbert spaces), whose vertical maps are $*$-homomorphisms and whose horizontal maps are bimodules [3]. In recent work, we show that such a double functor can be defined, reformulate the Haag-Kastler axioms in this framework, and provide examples of the construction [4].

In addition to this double-functorial formulation of AQFT, we discuss ongoing progress toward incorporating the von Neumann type classification into this framework. Type I algebras admit nonzero minimal projections, type II admit nonzero finite projections but no minimal ones, and type III admit no nonzero finite projections [5]. Building on Penneys' 2-categorical framework for tracial von Neumann algebras [6], we outline a tracial/semifinite refinement of the codomain designed for type I/type II phenomena. We also indicate how the type III case suggests a further refinement incorporating modular-theoretic structure, which we plan to develop in subsequent work.

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Constructing the category of quantum graphs

Andre Kornell

Joint work with: Bert Lindenhovius

Quantum graphs are structures that first arose in the problem of quantum error correction [2]. This talk will describe the closed symmetric monoidal category of quantum graphs that was recently introduced in [4]. The focus of the talk will be the construction of this category from the perspective of category theory and the relevance of this category to quantum information theory.

The symmetric monoidal category qGph of quantum graphs can be obtained from the complex numbers $\mathbb{C}$ by a sequence of category-theoretic constructions, where we view $\mathbb{C}$ as an Ab-enriched dagger category with a single object. This sequence of constructions passes through the Sup-enriched dagger category qRel of quantum sets and relations, which serves as an allegory-like setting for discrete quantum mathematics [3]. The structure of qRel yields definitions of both the category qGph of (discrete) quantum graphs and the category qGrp of (discrete) quantum groups via allegorical internalization.

The symmetric monoidal category qGph contains the symmetric monoidal category Gph as a full monoidal subcategory. Here, Gph consists of graphs in which loops are allowed and multiple edges are forbidden, and its monoidal product is the box product $G \,\square\, H$, for which $(g_1, h_1) \sim (g_2, h_2)$ if both $g_1 \sim g_2$ and $h_1 = h_2$ or both $g_1 = g_2$ and $h_1 \sim h_2$. The symmetric monoidal structure on qGph is defined analogously. We prove that Gph and qGph are closed by internalizing the same abstract argument in Rel and qRel, respectively. Furthermore, qGph is enriched over Gph, and the functor $\mathsf{qGph}(K_1, -)\colon \mathsf{qGph} \to \mathsf{Gph}$ has a full and faithful left adjoint, making Gph a coreflective monoidal subcategory of qGph.

The category qGph is connected to two topics in quantum information theory in the possibilistic regime. The first topic is zero-error transmission over a quantum channel. We associate with every quantum channel φ a morphism T_φ of qRel that retains its possibilistic data, so that the confusability quantum graph of φ has adjacency relation $T_\varphi^\dagger \circ T_\varphi$. The classical and quantum capacities of the quantum channel can then be defined in terms of this confusability quantum graph using the structure of qGph. We show that every finite quantum graph is a confusability quantum graph, answering a question of Daws [1, section 6.2]. From this perspective, the morphisms of qGph correspond to those quantum channels that respect this implicit confusability structure in a natural sense and that do not increase a variant of von Neumann entropy.

The second topic in quantum information theory that has a connection to qGph is quantum nonlocality. Graph homomorphisms games form a key class of examples of quantum nonlocality in the possibilistic regime. For finite simple graphs G and H, the (G, H)-homomorphism game is a nonlocal game that has a winning strategy iff the external hom graph $\mathsf{qGph}(G, H)$ is nonempty, i.e., not initial. However, when the players share entangled quantum systems, they may have a winning strategy even when $\mathsf{qGph}(G, H)$ is empty [5]. We show that the (G, H)-homomorphism game has such a winning quantum strategy iff the internal hom quantum graph $[G, H]$ is nonempty. Thus, the closed symmetric monoidal structure of qGph encodes the existence of winning quantum strategies in a natural way that directly generalizes the classical case.

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Cartesian Linearly Distributive Categories

Rose Kudzman-Blais

Joint work with: Jean-Simon Pacaud Lemay

Linearly distributive categories (LDC) were introduced by Cockett and Seely to provide alternative categorical semantics for multiplicative linear logic [2], generalizing $*$-autonomous categories, by taking multiplicative conjunction and disjunction as primitive notions. So briefly, a LDC is a category with two monoidal products, tensor $\otimes$ and par $\oplus$, whose interaction is mediated by *linear distributivities.* A *cartesian linearly distributive category* (CLDC) is a LDC whose tensor is the categorical product $\otimes = \times$ and par is the coproduct $\oplus = +$.

Two monoidal products is not unusual, perhaps the most well-know definition in this branch being the distributive category [1]. It was initially believed that LDCs were a weakening of distributive categories, more precisely that the notion of a CLDC and of a distributive category would coincide. This was later found out not to be the case [2]. Consequently, the study on CLDCs was not pursued further at the time.

With recent developments for and applications of LDCs, there has been renewed interest in CLDCs. In particular, the development of a *linearly distributive Fox theorem* [3] (presented at CT2025) lead to further investigation into the internal workings of CLDCs and a search for new examples.

In this talk, we will discuss various important structural properties of CLDCs which we've uncovered [4]. While the definition of a CLDC seems straightforward, it has become clear that linear distributivity between cartesian and cocartesian structures is distinctive, resulting in rather nuanced behavior within CLDCs. One such key observation is that in a CLDC, its terminal object is always preinitial and, dually, its initial object is always subterminal.

Moreover, we will discuss two key classes of examples: bounded distributive lattices and semi-additive categories. A CLDC must often fall into one of these two categories via collapse theorems. For example, a CLDC must be semi-additive if it either has *invertible* linear distributivities or if it is *isomix*. Additionally, by applying these collapse theorems, we revisit a previously assumed class of CLDCs, the Kleisli categories of exception monads of distributive categories, and show that they are not, in fact, CLDCs. That said, while these collapse theorem may seem to constrain the landscape of possible CLDCs, we will provide a Grothendieck construction to generate new examples of CLDCs, producing some which are neither bounded distributive lattices nor semi-additive categories.

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Orthogonal Factorization Systems for Double Categories

Matthew Kukla

Joint work with: C.B. Aberlé, Elena Caviglia, Rubén Maldonado, Luca Mesiti, Dorette Pronk, and Tanjona Ralaivaosoana

Orthogonal factorization systems in a 1-category provide a notion of images along arrows. Various factorization systems on the category of categories give rise to important notions of images for functors, such as replete images, essential images and full images. In general, orthogonal factorization systems reveal important properties of the structure of a category, and may assist in defining a functor on a category.

In this talk, we will generalize orthogonal factorization systems to the double-categorical setting. We call these *double orthogonal factorization systems* (DOFS); this will allow us to define images of double cells in the direction of tight arrows. Although a notion of factorization for double cells was introduced in [5], no orthogonality conditions are imposed. In order to obtain a notion of DOFS, we will consider normal pseudo-category objects in a suitable 2-category of orthogonal factorization systems. However, normal pseudo-category objects can be viewed as models of a limit sketch with two types of arrows: tight arrows for domain, codomain, and identity, and a loose arrow for composition. We consider category objects in various enhanced 2-categories of orthogonal factorization systems. Classically, a good way to capture and understand orthogonal factorization systems and morphisms between them is in terms of algebras for a particular 2-monad. A result of Coppey [3] shows that strict factorization systems can be viewed as strict algebras for the squaring 2-monad (sending any category to its category of squares). The full result characterizing orthogonal factorization systems with a chosen factorization as the pseudo-algebras for this 2-monad was established by Korostenski and Tholen in [4]. This leads us to four natural notions of morphism between categories with an OFS: the strict algebra maps preserve the chosen factorizations, the pseudo maps preserve both classes of morphisms, the lax maps preserve the right class, and oplax maps preserve the left class. This provides us with three enhanced 2-categories in which to define a corresponding notion of DOFS. We illustrate this with several examples in each category of algebras. Several common types of double categories, including double categories of spans, relations, and modules, are shown to admit double factorization system under our definition. Further examples of DOFSs will be presented by R. Maldonado in a subsequent talk.

Another key aspect of orthogonal factorization systems is their interaction with Grothendieck fibrations: any fibration gives rise to a *Cartesian factorization system* on its domain, whose left class consists of the vertical arrows in the fibration (and satisfies the 3-for-2 property) and right class consists of Cartesian arrows. This result can be generalized to show that any orthogonal factorization system on the base of a fibration can be lifted to the total category canonically. We show that the recently-developed theory of double fibrations in [2] interacts analogously with our notion of orthogonal factorizations on double categories, allowing a DOFS to be lifted along a double fibration.

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A Giraud-Conduché condition for T-categories

Steve Lack

Joint work with: Soichiro Fujii

Given a monad T on a category $\mathcal{E}$, Burroni introduced the notion of T*-category*, also known as multicategories, generalized multicategories, and T-multicategories. For suitable choices of $\mathcal{E}$ and T, this notion incluces multicategories in the sense of Lambek, internal categories in the sense of Ehresmann, topological spaces, and virtual double categories in the sense of Cruttwell and Shulman (earlier studied by Leinster under the name of fc-multicategories). The notion of T-category has received increasing interest in recent years and has been studied by too many people to list here.

Since the notion of T-category includes that of internal category and so in turn that of category, we can seek to generealize various categorical notions to the T-categorical context, either in general, or for particular choices of $\mathcal{E}$ and T. In earlier work, we defined a notion of *nerve* for a T-category, and studied local presentability and other such properties of the category (or 2-category) of T-categories. In this talk, we look at the situation where a T-category or T-functor is *powerful* (also known as exponentiable).

Recall that an object A of a category $\mathcal{K}$ with finite products is powerful when the functor $- \times A\colon \mathcal{K} \to \mathcal{K}$ has a right adjoint; thus all objects are powerful just when $\mathcal{K}$ is cartesian closed (has internal homs or mapping spaces). A morphism $A \to X$ is powerful when it is powerful as an object of the slice category $\mathcal{K}/X$; again, all morphisms are powerful when $\mathcal{K}$ is locally cartesian closed.

As is well-known, **Cat** is cartesian closed, but not locally cartesian closed. The powerful morphisms in **Cat** were characterized by Giraud and by Conduché, and they include both the fibrations and the opfibrations. We study the powerful T-categories and the powerful T-functors (morphisms of T-categories) under assumptions on $\mathcal{E}$ and T: the category $\mathcal{E}$ should be something like a topos, the functor T should be a parametric right adjoint (and so in particular preserve connected limits), and the multiplcation and unit of the monad should be cartesian natural transformations. Under these assumptions we give a sufficient condition for a T-category or T-functor to be powerful. Our condition agrees with that of Giraud-Conduché in the case of ordinary functors, and with the known generalization to internal functors. In the case where $\mathcal{E} = \textbf{Set}$ and T is the monoid monad, so that T-categories are multicategories in the sense of Lambek, we recover Pisani's result that the powerful multicategories are the promonoidal categories.

Local categories: a new framework for partiality

Marcello Lanfranchi

Joint work with: Jean-Simon Pacaud Lemay

Restriction categories offer a categorical framework for partiality. Restriction categories are now a well-established active field of research with a rich literature [1]. In this talk, we introduce three new categorical theories for partiality: local categories, partial categories, and inclusion categories. The objects of a **local category** are partially accessible resources, and morphisms are processes between these resources. In a **partial category**, partiality is addressed via two operators, **restriction** and **contraction**, which control the domain of definition of a morphism. Finally, an **inclusion category** is a category equipped with a family of monics which axiomatize the **inclusions** between sets.

Our main result shows that restriction categories are 2-equivalent to local categories, that partial categories are 2-equivalent to inclusion categories, and that both restriction/local categories are 2-equivalent to bounded partial/inclusion categories. In particular, given a restriction category $\mathbb{X}$, we construct a local category $\mathsf{L}[\mathbb{X}]$ by taking the subcategory of total maps of the split restriction idempotent completion of $\mathbb{X}$ and we show that this correspondence is in fact a 2-equivalence. Furthermore, we show that every local category carries two operators which satisfy the axioms of a partial category and, from these two operators, we construct a special family of monics that makes the category into an inclusion category. Finally, we prove that every partial and inclusion category is, in fact, a local category provided it satisfies an extra assumption that we call **boundedness**.

Our result offers four equivalent ways to describe partiality: on morphisms, via restriction categories; on objects, with local categories; operationally, with partial categories; and via inclusions, with inclusion categories.

Moreover, we also consider our equivalence on an important class of restriction categories: inverse categories [4]. In this case, the equivalence between inverse categories and inverse local categories is a generalization of the celebrated ESN theorem for inverse semigroups [3], and recapturing some of the constructions of DeWolf and Pronk [2].

Paper. `https://arxiv.org/abs/2512.03371`

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From Analysis to Stable Homotopy Theory via lax-idempotent Monads

Georg Lehner

Joint work with: Fernando Abellán, Thomas Blom

Recent progress in the theory of dualizable categories due to Efimov, Nikolaus, Clausen among others have made it possible to use analytic/operator-theoretic techniques in the setting of stable ∞-categories. This theory provides a promising framework for understanding questions in geometric topology, such as assembly conjectures like the Borel, Novikov and Farrell-Jones conjectures. It was observed that dualizable categories share many formal similarities with compact Hausdorff spaces - In particular, variants of the Tychonoff Theorem and Urysohn Lemma exist for dualizable categories.

We explore these similarities using the notion of a lax-idempotent monad, also called KZ-monad, on a 2-category. Every lax-idempotent monad has a corresponding category of *continuous algebras*. Examples of these are abundant: If one starts with the category of small stable ∞-categories and the monad given by Ind, one obtains the category of dualizable categories. If one starts with the category of distributive lattices, one obtains stably compact spaces. The category of symmetric monoidal categories with the envelope monad Env leads to operads. Due to the formal nature of this construction, several highly non-trivial results, such as Aoki's Sheaves-Smashing spectrum adjunction, and Verdier Duality for locally compact Hausdorff spaces, can be understood using 2-categorical techniques, and allow straightforward generalizations.

Formal ∞-category theory of relative and simplicial categories, and more general enriched categories.

Giuseppe Leoncini

Several models of $(\infty, 1)$-categories exist [3], making precise the idea of a category with a homotopy-coherent weak enrichment in spaces (considered up to weak homotopy equivalence). Two different axiomatizations of the homotopy theory of $(\infty, 1)$-categories have been given by Toën in [5], and by Barwick and Schommer-Pries in [2]. Knowing that two models are equivalent in the sense that they present the same homotopy theory is not sufficient, however, to conclude that they produce "the same category theory" and that categorical results proven using one particular model can be assumed to apply to all the others. A breakthrough towards model independence has been achieved by Riehl and Verity in a series of works culminating in [4]. Their framework applies to various models of $(\infty, 1)$-categories, namely quasicategories, complete Segal spaces, Segal categories, and saturated 1-trivial weak 1-complicial sets. There are two notable exceptions: simplicially enriched categories and relative categories [1]. In this talk, I will show how to fit these into the picture. These two models are important for a variety of reasons: many fundamental examples of $(\infty, 1)$-categories arise most naturally as relative categories (for example, those underlying a model category) or as simplicially enriched categories (for example, the ∞-category of spaces itself); moreover, they are a convenient setting for explicit computations via homotopy (co)limits. What matters for the purpose of developing $(\infty, 1)$-category theory synthetically as in [4] is the existence of a structure called a proarrow equipment [6] on a suitably defined homotopy 2-category obtained from a model; if two models produce equivalent proarrow equipments, then, in essence, they can be used interchangeably for doing $(\infty, 1)$-category theory. The methods used in [4] to establish the existence and equivalence of the various proarrow equipments do not apply to simplicial categories and relative categories, hence one must proceed in an ad hoc way. In the final part of the talk I will outline how to generalize this result from simplicially enriched categories to categories enriched in a monoidal model category $(\mathcal{V}, \mu)$, thus producing a way to encode the theory of ∞-categories enriched in (the underlying ∞-category of) the model category $(\mathcal{V}, \mu)$.

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Characterizing Tangent Display Maps via Linear Assignments

Ruiliang Li

Tangent display maps, introduced by Cruttwell and Lanfranchi [2], provide the abstract analogue of submersions inside a tangent category [1]: they single out those morphisms along which one can pull back differential bundles compatibly with connection data. In algebraic examples the tangent functor is typically far from left exact, so deciding whether a given map is "display" becomes a genuinely algebraic problem.

We work in the tangent categories constructed from a *linear assignment* L in the sense of Ikonicoff–Lemay–Van der Linden [3]. Such an L is a product-preserving endofunctor equipped with a natural isomorphism $\nu\colon LL \Rightarrow L$, and it induces $T(X) = X \times L(X)$ and $T(f) = f \times L(f)$. A key feature is that the iterates admit explicit decompositions

$$T^n(X) \cong X \times (L(X))^{2^n - 1} \qquad (n \geq 0),$$

built functorially from the product comparison maps for L and the idempotence isomorphisms ν.

Given a pullback square

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle q} \\ N & \xrightarrow[f]{} & M, \end{array}$$

there is a canonical Beck–Chevalley comparison morphism $\beta_{f,q}\colon L(P) \to L(N) \times_{L(M)} L(E)$. Our main result shows that, for linear-assignment tangent functors, the *infinite* requirement that a pullback be preserved by all iterates of T collapses to a *single* Beck–Chevalley check at the level of L.

Theorem. Let $(\mathcal{C}, T)$ be the tangent category induced by a linear assignment L on a finitely complete category $\mathcal{C}$. For any pullback square as above, the following are equivalent: (i) it is a $\mathbb{T}$-pullback (i.e. it is preserved by every iterate T^m, $m \geq 1$); (ii) it is preserved by T; (iii) the map $\beta_{f,q}$ is an isomorphism. Moreover, under $T^m(X) \cong X \times (L(X))^{2^m - 1}$, the T^m-image of the square is the product of the original pullback square with $(\beta_{f,q})^{2^m - 1}$.

In the strongly unital (hence semi-abelian) situation of [3], L is commutativization/abelianization (e.g. $T(G) = G \times G^{\mathrm{ab}}$ in **Grp**). Taking f to be the zero map identifies $\beta_{0,q}$ with the canonical comparison $L(\ker q) \to \ker(Lq)$, giving an immediate obstruction for regular epimorphisms to be tangent display maps. We use the explicit formulas for the iterates T^n to control the remaining Beck–Chevalley conditions in the definition, and we extract pullback-stable families of regular epimorphisms satisfying them, providing concrete algebraic analogues of submersions.

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Pointed Univalence

Yufeng Li

Joint work with: Krzysztof Kapulkin

Voevodsky's Univalence Axiom is perhaps the most fundamental logical principle introduced in the 21st century. A universe of (small) types satisfies the univalence axiom if, for any two types in that universe, their type of equalities (or identifications) is equivalent to the type of equivalences between them. Thus, for example, two logically equivalent propositions or bijective sets can be treated as equal, and hence substituted for each other in all contexts. While convenient and often assumed in informal mathematical practice, the univalence axiom requires formal justification. The first such justification comes from Voevodsky's celebrated simplicial model [2]; the model that not only satisfies, but in fact inspired, the univalence axiom.

In [2], the goal is to build a single model of homotopy type theory (in simplicial sets), and when it comes to verifying univalence, the authors do just that: they translate the statement into a statement about simplicial sets and check it directly. In contrast, in [1], we make two further contributions to categorical univalent type theory. First, we provide a more general semantic treatment of univalence at the level of Voevodsky's universe categories [5] by rephrasing the condition in a way that is easy to check across various models. We also aim to avoid using the internal language formulations which, while elegant, might occasionally conceal important details, possibly leading to incomplete arguments. Altogether, the upshot of our formulation is that, when verifying univalence in a universe category model, one does not need to work with syntax at all.

Interestingly, to arrive at a formulation of univalence that can be easily verified in a universe category, we actually give a stronger statement than the one commonly used and given in the HoTT Book [4], which brings us our second contribution. As mentioned before, univalence traditionally formulated by saying that a certain map from the identity type between two types in a universe to the type of equivalences between them is itself an equivalence. Our strengthening requires that the homotopy inverse of this map sends the identity equivalence to the reflexivity term. To differentiate the two, we call the version found in [4] *book univalence* and our new version *pointed univalence*, which is based on a lifting condition. While book univalence asks for a certain commutative square to admit a diagonal filler making only the lower triangle commute, pointed univalence requires that both resulting triangles commute. As such, pointed univalence is natural to verify in models coming from Quillen model categories, where fillers make both triangles commute. Furthermore, since maps from the left class to fibrations axiomatize pattern matching, our pointed univalence is also computationally desirable, as it justifies performing pattern matching on equivalences. Thus, we believe this strengthening constitutes a new notion of independent interest with strong semantic justification.

Our semantic study of pointed univalence includes studying closure properties of models under two fundamental constructions: Artin–Wraith gluing and inverse diagrams. For book univalence, these were verified in the seminal work of Shulman [3], and we provide the "pointed counterpart" of his results.

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Triadic Equivalence of Regular Lawvere Theory

Chun-Yu Lin

Regular Lawvere theories, introduced by S. Szawiel and M. Zawadowski, provide a categorical account of regular varieties in universal algebra [1]. They show that the categories of regular Lawvere theories **RegLT**, semi-analytic monads **SanMnd**, and regular operads **RegOp** are equivalent. In this talk, we investigate two directions in which this triadic equivalence can be generalized.

Our first generalization is to the $\mathcal{V}$-enriched setting, where $\mathcal{V}$ is a locally presentable cosmos in the sense of B'enabou. We introduce regular Lawvere $\mathcal{V}$-theories $\mathcal{V}$**-RegLT** and semi-analytic $\mathcal{V}$-monads $\mathcal{V}$**-SanMnd**, and establish equivalence between them following [3]. We then define regular $\mathcal{V}$-operads $\mathcal{V}$**-RegOp** and prove, using [2], that $\mathcal{V}$**-RegOp** is equivalent to $\mathcal{V}$**-SanMnd**. Altogether, we obtain

$$\mathcal{V}\textbf{-RegLT} \simeq \mathcal{V}\textbf{-SanMnd} \simeq \mathcal{V}\textbf{-RegOp}.$$

As a corollary, taking $\mathcal{V}$ to be self-enriched yields a one-to-one correspondence between symmetric monoidal monads $\mathsf{S}\mathcal{V}$ on $\mathcal{V}$ and $\mathcal{V}$-enriched monads $\mathsf{M}\mathcal{V}$ on $\mathcal{V}$ [7]. Combining this with the equivalence between commutative strong monads and symmetric monoidal monads, we deduce that $\mathsf{S}_{\mathcal{V}}$ corresponds to a commutative regular Lawvere $\mathcal{V}$-theory, and equivalently to a regular $\mathcal{V}$-operad equipped with the Boardman–Vogt tensor product [5].

Our second generalization replaces Lawvere theories by PROPs, motivated by the fact that a Lawvere theory can be viewed as a cartesian PROP. We define a notion of regular PROP **rProp** and prove its equivalence with regular colored operads **cRegOp** via the adjunction studied in [6]. Moreover, we show that **rProp** is equivalent to the category of semi-analytic monads on polygraphs $\mathbf{SanMnd_p}$ [4]. Consequently, we obtain

$$\mathbf{rProp} \simeq \mathbf{cRegOp} \simeq \mathbf{SanMnd_p}$$

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Constructing Canonical Calculi

Gabriele Lobbia

Joint work with: Keegan J. Flood, Giacomo Tendas

In noncommutative geometry, one typically equips an associative algebra A with further structure such that it can then be regarded as encoding a “noncommutative space”. A general axiomatic approach is via the notion of a *differential calculus* (also sometimes called *exterior algebra*), introduced by Woronowicz [1].

Given an associative algebra A, there is a canonical choice of first order differential calculus, namely the universal first order differential calculus, obtained as the kernel of the multiplication on A. It is widely studied although it is often regarded to be of considerably less geometric interest than some classical commutative examples (e.g. de Rham forms for C^∞-rings or Kähler differentials for commutative algebras). Therefore, a standard practice in the noncommutative setting is to consider additional structures (bimodule relations, covariance conditions, etc.) in order to obtain more geometrically interesting notions of differential calculus. Unfortunately, such an approach is, in general, insufficient to yield existence or uniqueness of a compatible notion of form, and such procedures are typically not functorial. The foundational work on differential calculi [1, p. 126] specifically remarks upon the “unpleasant contrast” with the classical case which this lack of functoriality results in (there Woronowicz is specifically referring to the case of quantum groups).

In this work, we aim to address this problem treating the geometry of a category $\mathcal{E}$ as a *relative* notion: it will emerge when $\mathcal{E}$ is viewed in relation to a category of monoids $\mathrm{Mon}(\mathcal{V})$ via a faithful isofibration $(-)_0 \colon \mathcal{E} \to \mathrm{Mon}(\mathcal{V})$, where $\mathcal{V}$ is a monoidal additive category. In this setting, we can define a notion of *bimodule* category for objects of $\mathcal{E}$, which generalises the case of bimodules over a monoid in $\mathcal{V}$, through the following pullback.

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{E}) & \xrightarrow{K} & \mathrm{Mod}(\mathcal{V}) \\ {\scriptstyle U'}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle U} \\ \mathcal{E} & \xrightarrow[(-)_0]{} & \mathrm{Mon}(\mathcal{V}) \end{array}$$

We start by generalising the notion of first order differential calculi to the setting of monoids internal to a monoidal additive category $\mathcal{V}$ and show that the standard results concerning first order differential calculi extend to this broader setting. Then, we establish sufficient conditions on the faithful isofibration $(-)_0$ such that $\mathcal{E}$ admits a canonical functor $\mathrm{Calc}^1_{\mathcal{E}}$ to the category $\mathrm{Calc}^1(\mathcal{V})$ of first order differential calculi in $\mathcal{V}$. Generalising the procedure of extending a first order differential calculus to its maximal prolongation to this setting, we obtain a canonical *de Rham* functor from $\mathcal{E}$ to the category $\mathrm{Calc}^\bullet(\mathcal{V})$ of differential calculi in $\mathcal{V}$.

$$\mathcal{E} \xrightarrow{\mathrm{Calc}^1_{\mathcal{E}}} \mathrm{Calc}^1(\mathcal{V}) \xrightarrow{(-)_{\max}} \mathrm{Calc}^\bullet(\mathcal{V})$$

This yields a simultaneous generalisation of the de Rham complex on C^∞-rings, the Kähler differentials on commutative algebras, and the universal differential calculus on associative algebras. As a consequence, such categories $\mathcal{E}$ admit natural analogues of the notions of smooth map and diffeomorphism, as well as a functorial de Rham theory. Moreover, whenever two such faithful isofibrations to $\mathrm{Mon}(\mathcal{V})$ factor suitably, their corresponding de Rham functors are related via a comparison map.

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Locally presentable categories over a base, S-sorted limit theories, and cartesian first-order theories

Rory Lucyshyn-Wright

Joint work with: Andrew Krenz, Jason Parker

The usual Lawvere theories are intrinsically single-sorted but admit both an S*-sorted* generalization and an *unsorted* analogue (namely small categories with finite products). The situation in the existing literature on locally presentable categories is rather different: α*-limit theories* are small categories with α-small limits, seen as unsorted theories, and they correspond under the Gabriel-Ulmer duality to locally α-presentable (LαP) categories, while a set of sorts S appears only in certain related syntactic theories: Indeed, an LαP category $\mathcal{A}$ is equivalently the category of models of

(1) an S-sorted α-ary *cartesian theory* [1, 2, 3] for some set S, and

(2) an S'-sorted α-ary *essentially algebraic (e.a.) theory* [2, 4] for some set S'

but in general the sets of sorts S and S' are distinct, and the resulting 'carrier' functors $\mathcal{A} \to \mathrm{Set}^S$ and $\mathcal{A} \to \mathrm{Set}^{S'}$ have different properties, with the latter conservative and the former in general not.

In this talk, we fix a set S and develop two sharper correspondences that refine the above, namely correspondences between each of the following triples of concepts, after first defining the italicized terms:

I. (i) S*-sorted* α*-limit theories*, (ii) S*-sorted L*α*P categories*, (iii) S-sorted α-ary cartesian theories,

II. (i) *e.a.* S*-sorted* α*-limit theories*, (ii) *e.a.* S*-sorted L*α*P categories*, (iii) S-sorted α-ary e.a. theories,

where we write *e.a.* as an abbreviation of *essentially algebraic*. Explicitly, (I.i) and (II.i) are defined as α-limit theories equipped with suitable morphisms of α-limit theories $\tau : (\mathrm{Set}^S)^{\mathrm{op}}_\alpha \to \mathcal{T}$, and (I.ii) and (II.ii) are LαP categories $\mathcal{A}$ equipped with suitable functors $\mathcal{A} \to \mathrm{Set}^S$ that are necessarily faithful. More generally, we define special classes of 1-cells of LαP categories $\mathcal{A} \to \mathcal{C}$ (called *single-sorted* and *e.a.* 1-cells) that specialize to (I-II.ii) by taking $\mathcal{C} = \mathrm{Set}^S$ and correspond under the Gabriel-Ulmer duality to special classes of morphisms of α-limit theories.

These results have the advantage of providing a more nuanced 'dictionary' that relates distinct notions of logical theory to corresponding distinct categorical concepts. Moreover, by regarding categories of models of cartesian theories as concrete categories over Set^S, we are able to provide categorical formulations of logical aspects of such categories that involve operations and relations, whose *arities* are necessarily objects of Set^S. We show that every α-presentable object A of such a category admits a *presentation* $A \cong \langle x \mid \varphi \rangle$ in terms a family of *epigenerators* x and an α-ary cartesian formula φ. We use our results to shed light on questions of whether compactness and completeness theorems are available in the setting of α-ary cartesian logic (recalling that they are in general unavailable for α-ary first-order theories). In turn, we apply these results to prove further results on the question of whether every faithful 1-cell of LαP categories is single-sorted.

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An Invitation to Geometric Type Theory

Owen Lynch

Joint work with: David Jaz Myers

Geometric Type Theory is a conjectured type theory that would provide an internal language for the 2-category of topoi [6]. In this talk, we give a direct construction of a category with familes (CwF, [1]) permitting interpretation of such a type theory. This is inspired by a close reading of the Elephant [3]; Johnstone defines a geometric theory over a fixed topos $\mathcal{S}$ to be a 2-functor $\mathcal{T}$ from $(\mathsf{BTop}/\mathcal{S})^{\mathrm{op}}$ to Cat that is built by applying a sequence of PIE limits to the object classifier $T_0(\mathcal{E}) = \mathcal{E}$. From this, we define a CwF:

CwF sort	**Interpretation**
Context Γ	Geometric theory $\Gamma\colon \mathsf{Top}^{\mathrm{op}} \to \mathsf{Cat}$
Substitution $\Delta \vdash \gamma\colon \Gamma$	2-natural transformation $\gamma\colon \Delta \Rightarrow \Gamma$
Type $\Gamma \vdash A\,\mathsf{type}$	Geometric theory over Γ, that is $A\colon \int \Gamma \to \mathsf{Cat}$
Term $\Gamma \vdash a\colon A$	Natural section $a(\mathcal{E}, M) \in A(\mathcal{E}, M)$ for $(\mathcal{E}, M) \in \int \Gamma$

Note that $\int \Gamma$ is equivalent to topoi sliced over the classifying topos for Γ, by representability of geometric theories. This CwF then supports a rich variety of type formers.

Example 1. There is a type $\Gamma \vdash \mathsf{Sort}\,\mathsf{type}$ that is is given semantics by the object classifier $\int \Gamma \to \mathsf{Cat}$ defined by $(\mathcal{E}, M) \mapsto \mathcal{E}$. Given $\Gamma \vdash A\colon \mathsf{Sort}$, we have another type $\Gamma \vdash \mathsf{Elt}\,A\,\mathsf{type}$ defined by $(\mathcal{E}, M) \mapsto \mathcal{E}(1, A(\mathcal{E}, M))$, so Sort acts as a "universe of small types". The object classifier Sort serves a dual role: it helps us build up theories, but additionally when we work internally to Sort in a fixed context Γ we get the positive fragment of the internal language for the classifying topos for Γ.

Example 2. There is a "Π-type with small codomain" $(x\colon A) \to B$ defined for $\Gamma \vdash A\colon \mathsf{Sort}$ and $\Gamma, x\colon \mathsf{Elt}\,A \vdash B\,\mathsf{type}$. The interpretation of $(x\colon A) \to B$ on $(\mathcal{E}, M) \in \int \Gamma$ is given by interpreting $B[x]$ in the slice topos $\mathcal{E}/A(\mathcal{E}, M)$, where x is the global element of A in $\mathcal{E}/A(\mathcal{E}, M)$ given by the identity.

We also have Σ-types, both globally and within the Sort universe.

In fact, this construction scales beyond just topoi/geometric theories. From Garner and Lack [2], we know that various fragments of geometric logic correspond to sub 2-monads of $\mathsf{Psh}\colon \mathsf{Lex} \to \mathsf{Lex}$ given by completion under various classes of weighted colimits. Type theoretically, this corresponds to allowing different kinds of (strictly positive) type formers. For instance, regular logic corresponds to adding propositional truncation, with the elimination principle as given in the HoTT book [4]. Especially interesting for us is the setting of arithmetic universes [5], which interprets quotient inductive inductive types.

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Double Orthogonal Factorization Systems II

Rubén Maldonado

Joint work with: Dorette Pronk, Luca Mesiti, Elena Caviglia, Matthew Kukla, C.B. Aberle, Tanjona Ralaivaosaona

This talk continues the presentation by Matthew Kukla on double orthogonal factorization systems (DOFS) by focusing on extensions of ordinary orthogonal factorization systems (OFS) to the double categorical setting [1, Section 5]. Given a double category whose category of objects and arrows carries an orthogonal factorization system, one may ask when this structure extends to a DOFS.

First, when the category of proarrows and double cells also carries an OFS, we present conditions under which chosen factorizations can be adjusted so that the source, target, and unit functors become strict morphisms of OFS, yielding a genuine double orthogonal factorization system. Using the notions of restrictions (cartesian cells) and extensions (opcartesian cells) in a double category [2] [3], we then give properties that ensure the existence of at least one DOFS extending a given OFS on the arrow category, and show that such a DOFS compares to any other DOFS sharing the same underlying OFS via a unique morphism of DOFS. As a particular case, in any fibrant double category [4], there exist a canonical initial and a canonical terminal DOFS over any given OFS on the arrows, bracketing all others. We illustrate these constructions with two concrete examples: the double category of relations enriched over a quantale, and the double category of spans.

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Coherence for pseudo commutative 2-monads

Diego Manco

Under mild conditions, the category of algebras (and strict maps) $T\text{-alg}_s$ of a commutative monad on a symmetric monoidal closed category $T\colon \mathcal{V} \to \mathcal{V}$ is symmetric monoidal closed [4]. By merging the concept of commutative monad with that of 2-monad one is led to the definition of a strictly commutative 2-monad over a symmetric monoidal 2-category. With the motivation of studying some 2-monads $T\colon \mathcal{K} \to \mathcal{K}$ which are not strictly commutative, such as the 2-monad in **Cat** for symmetric strict monoidal categories, as well as their categories of algebras and pseudo maps $T\text{-}alg$, Hyland and Power defined the concept of a pseudo commutative 2-monad $T\colon \mathcal{K} \to \mathcal{K}$ on a symmetric monoidal 2-category [3]. These are 2-monads which are strictly commutative up to coherent isomorphisms in a precise sense. For such 2-monads $T\colon \mathcal{K} \to \mathcal{K}$, they proved that $T\text{-}alg$ can be enhanced to a **Cat**-enriched non-symmetric multicategory, and that when T satisfies the extra condition of being symmetric, $T\text{-}alg$ is a symmetric **Cat**-enriched multicategory [3].

Our first result is that when T is a symmetric, pseudo commutative 2-monad, the free algebra functor $T\colon \mathcal{K} \to T\text{-}alg$ can be enhanced to a non-symmetric **Cat**-enriched multifunctor [7]. This **Cat**-enriched multifunctor fails to preserve the action of the symmetric group on multilinear maps by swapping inputs, but it does so up to coherent isomorphisms. Such **Cat**-enriched multifunctors are called pseudo symmetric and they were defined by Yau, who proved that inverse K-theory gives one example [8]. Our second result is that when T is a symmetric pseudo commutative 2-monad, the **Cat**-enriched multifunctor $T\colon \mathcal{K} \to T\text{-}alg$ is pseudo symmetric [7]. By using results from [6], we can rigidify $T\colon \mathcal{K} \to T\text{-}alg$ to get a symmetric **Cat**-enriched multifunctor $T\colon \mathcal{K} \times E\Sigma_* \to T\text{-}alg$, where $E\Sigma_*$ is the categorical Barrat-Eccles operad. This can be considered as a coherence theorem for symmetric pseudo commutative 2-monads. Our proof also implies a coherence theorem conjectured by Hyland and Power [3] for the case when T is not symmetric.

Our results will find applications in Algebraic K-theory since symmetric pseudo commutative 2-operads [1, 2], 2-operads whose associated 2-monads are symmetric pseudo commutative, are used to parameterize the input of several equivariant algebraic K-theory constructions [2, 9]. They also apply to KZ 2-monads [5], which include 2-monads whose algebras are categories with a given class of colimits [5].

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Enriched mapping spaces via necklaces

Arne Mertens

Joint work with: Giuseppe Leoncini, Wendy Lowen

Necklaces are combinatorial gadgets originally introduced by Baues (who called them 'cellular strings') and popularised by Dugger-Spivak in their study of the mapping spaces of quasi-categories [2]. A *necklace* is a simplicial set built from a sequence of standard sequences that are glued at their endpoints, e.g.

$$T = \Delta^3 \vee \Delta^2 \vee \Delta^1 \vee \Delta^3$$

The category $\mathcal{N}ec$ of necklaces has many desirable properties. It is a Reedy category with a compatible monoidal structure [1]. Moreover it is a test category, hence the category $NSet$ of necklicial sets $\mathcal{N}ec^{op} \to Set$ models homotopy types. Consider the category $NCat$ of categories enriched in $NSet$. Then there exists a fully faithful left-adjoint $SSet \hookrightarrow NCat$ [3] from simplicial sets into necklicial categories. This fact was (implicitly) used in [2] to construct alternative models for the left-adjoint of the homotopy coherent nerve. In this talk I will explain how this result may be generalised to left-adjoints of other nerves.

Let $(\mathcal{W}, \otimes, I)$ be a monoidal simplicial model category. In [4], I provide a general procedure for constructing nerve functors $N^D : \mathcal{W}Cat \to SSet$ induced by a strong monoidal diagram $D : \mathcal{N}ec \to \mathcal{W}$, and give conditions for when its left-adjoint L^D may be described explicitly. In work in preparation I show, under some compatibility conditions between $\mathcal{W}$ and D:

Theorem. *Let X be a quasi-category with $a, b \in X_0$. Then there is a zig-zag of weak equivalences in $\mathcal{W}$:*

$$L^D(X)(a, b) \leftrightsquigarrow Map_X(a, b)$$

between the hom-object of $L^D(X)$ and the mapping space of X, considered as an object of $\mathcal{W}$.

This includes the left-adjoints of the homotopy-coherent, cubical and differential graded nerves. I will give a sketch of the proof, which is based on [2]. The crucial step is a surprisingly elegent combinatorial argument for necklaces.

This result fits into a broader project joint with Wendy Lowen [3], where we put forth the category of *quasi-categories in a monoidal category* $\mathcal{V}$ as a model for $S\mathcal{V}$-enriched categories with a homotopically well-behaved monoidal structure. When $\mathcal{V} = Set$, this recovers the classical comparison of quasi-categories with simplicial categories. Time permitting, I will expand on this and explain work in preparation on the case where $\mathcal{V} = Mod(k)$, the category of modules over a commutative ring k. I construct a monoidal cofibration category $S_\otimes Mod(k)_{cof}$, the fibrant objects of which are in particular quasi-categories in $Mod(k)$. Moreover, the above theorem allows to show an equivalence of homotopy categories

$$Ho(dgCat_{\geq 0}) \cong Ho(S_\otimes Mod(k)_{cof})$$

with connective dg-categories (equivalently $SMod(k)$-enriched categories by the Dold-Kan correspondence).

Finally, in joint work with Giuseppe Leoncini, we are developing the case where $\mathcal{V}$ is cartesian monoidal and locally connected. Time permitting, I will explain this work in further detail as well.

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Monoids, Monoidal Grothendieck Construction and Clifford Semigroups

Luca Mesiti

Joint work with: Elena Caviglia, Peter F. Faul, Graham Manuell

The Grothendieck construction exhibits one of the most fundamental equivalences in category theory. It gives the correspondence between Grothendieck fibrations and indexed categories, allowing one to freely move between the two worlds and enjoy the perks of both. In the last years, the theory of the Grothendieck construction has been actively expanded and generalized to broader settings, as well as successfully applied to both category theory itself and most of the areas of mathematics. Notably, Joe Moeller and Christina Vasilakopoulou extended the Grothendieck construction to the monoidal setting, motivated by abundant examples in algebra, dynamical systems, graphs and networks. They established an equivalence between monoidal fibrations and lax monoidal pseudofunctors into Cat.

In this talk, we present new categorical results on the monoidal Grothendieck construction as well their applications to the theory of Clifford semigroups and inverse semirings. Clifford semigroups are semigroups equipped with a weak form of inverses. In the commutative case, they are also known as inverse semigroups. They encompass a much broader set of examples, including partial bijections and semilattices. Inverse semirings are then semirings whose additive monoid is an inverse semigroup. A remarkable example is given by bounded polynomials.

We present what happens when taking monoids in the monoidal Grothendieck construction. We prove that monoids in the monoidal total category given by the monoidal Grothendieck construction precisely correspond to the Grothendieck construction of a pseudofunctor that takes monoids in the fibres. The monoids in the fibres are here considered with respect to a structure of monoidal category that is induced from the starting monoidal indexed category.

We then apply this result and the monoidal Grothendieck construction to categorically capture Clifford semigroups and inverse semirings. We show that Clifford semigroups equivalently correspond to discrete fibrations in groups over a meet semilattice. We then prove that, exactly as discrete fibrations can be collected into a Grothendieck fibration over Cat, the category of Clifford semigroups can be obtained as a monoidal Grothendieck construction over the category of semilattices. This extends the Structure Theorem for Clifford semigroups, giving rise to new applications in semigroup theory. In particular, we obtain results on factorization systems for Clifford semigroups.

Finally, we apply our result on monoids in the monoidal Grothendieck construction to the monoidal total category of inverse semigroups, in the commutative case. We obtain that inverse semirings equivalently correspond to taking monoids in the fibres, which are functor categories into the category Ab of abelian groups. We prove that, surprisingly, the induced monoidal structure on the fibres is precisely that of the Day convolution.

Classifying certain group extensions in HoTT

Owen Milner

Homotopy type theory is a formal language for reasoning about spaces [6]. It has semantics in higher toposes [5]. One family of spaces which are of particular interest in contemporary (higher) category theory are the so-called higher groups. A higher group is a space X equipped with a *delooping*, that is, a further space Y and an equivalence $X \simeq (\Omega Y)$ where ΩY is the loop-space of Y. In the setting of homotopy type theory, a particular class of higher groups, the *central* higher groups, was introduced by [1]. The central higher groups have many remarkable properties: for example, their deloopings are themselves central higher groups – which implies that the central higher groups can be delooped infinitely many times.

In this talk I will present a classification theorem for certain higher group extensions: extensions of truncated groups by central groups, in the setting of homotopy type theory. This builds on the work of Myers and Yasser on higher Schreier theory [3], and, as a special case, recovers a variant of the classical classification of 2-groups by their Postnikov invariants due in different forms to Mac Lane-Whitehead, and Sính [2, 4].

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Lyapunov Stability of Coalgebras

Joe Moeller

Joint work with: Aaron Ames, Sébastien Mattenet, and Paulo Tabuada

Stability of an equilibrium point is a central notion in the study of dynamical systems. In practice, it can be infeasible to check the stability of an equilibrium directly. Lyapunov's insight was that stability can instead be certified by the existence of an auxiliary object, a function which is positive definite relative to the equilibrium and decreases along the dynamics. Such a function is called a *Lyapunov function* for the system.

In this talk, we present a categorical formulation of Lyapunov's theory expressed in the language of endofunctor coalgebras. Working internal to a locally thin bicategory equipped with a monoid object representing time, we define equilibria, morphisms that are positive definite relative to equilibria, and stability of equilibria in purely categorical terms [1]. Within this setting, Lyapunov's theorem says that the existence of a positive definite lax coalgebra homomorphism implies stability of the equilibrium, and admits a simple proof via a pasting diagram argument [2].

The framework is sufficiently general to recover the classical Lyapunov theory for continuous-time dynamical systems, as well as its discrete-time counterpart. It also yields new Lyapunov-type stability results for labeled transition systems and Markov kernels. The theory extends to hybrid systems, in which continuous evolution interacts with discrete transitions [3]. We give an endofunctor on a category of charts in the sense of Myers [4] of which hybrid systems can be encoded as coalgebras. The framework subsumes several Lyapunov-type results for hybrid systems from the literature.

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Dual Equipments

J. Robert Morissette

The significance of fibrations to categorical logic is well-understood [2]. Elementary existential fibrations (EEFs), for example, model specifications of *regular* logic, the fragment of first-order logic whose logical operations consist only of $\top, \wedge$, and $\exists$.

In [3], Nasu established an equivalence between EEFs and a flavour of *equipment*, building on a construction from Shulman in [4], effectively bringing the logical connections associated with the fibrations into the double-categorical realm. Equipments are double categories for which the data of the tight structure is effectively encoded as part of the loose structure, in the form of *companions* and *conjoints* associated to each tight morphism. A typical example is the equipment $\mathbb{R}\mathrm{el}$, whose objects are sets, tight morphisms $f : A \to B$ are functions, loose morphisms $R : A \nrightarrow C$ are relations $R \subseteq A \times C$ (composed in the usual way), and there is a unique cell

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle R}\downarrow & \alpha & \downarrow{\scriptstyle S} \\ C & \xrightarrow[g]{} & D \end{array}$$

iff the implication $(a, c) \in R \Rightarrow \big(f(a), g(c)\big) \in S$ holds. $\mathbb{R}\mathrm{el}$ is an equipment, in which the companion of a tight morphism $f : A \to B$ is its graph $f_* = \{(a, b) \,|\, f(x) = y\} \subseteq A \times B$, and the conjoint f^* of f is the opposite relation $f^* \subseteq B \times A$. The fibration underlying $\mathbb{R}\mathrm{el}$ is $Sub(Set)$, the subobject fibration on Set.

A general goal of ours is to extend Nasu's correspondence to lift more logically significant aspects of fibrations into double categories to study them there, in order to leverage the additional expressive power of double categories. In this talk, our main focus will be on the dual fibration construction. Given any fibration $p : \mathcal{E} \to \mathcal{B}$, one may define the *dual fibration* of p, denoted $p^\bullet : \mathcal{E}^\bullet \to \mathcal{B}$ over the same base category, where $\mathcal{E}^\bullet$ is obtained by taking fibrewise opposites in $\mathcal{E}$. According to Nasu's result, in order for $p^\bullet$ to correspond to an equipment, which we denote $\mathbb{B}il(p^\bullet)$, it must be elementary existential, meaning that p must satisfy the duals of these properties to begin with. In this case, one could say that $\mathbb{B}il(p^\bullet)$ is the *dual equipment* of the equipment $\mathbb{B}il(p)$ associated with p. In this talk, we will characterize these properties in both logical and double-categorical terms, and show the functorial semantics for *dualizable* equipments.

Our motivating example for dual equipments is the double category $\mathbb{R}\mathrm{el}^\bullet$, which has the same objects, tight and loose morphisms as $\mathbb{R}\mathrm{el}$, but where the composition of two relations $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$ is given by $R \oplus S := \{(a, c) \,|\, \forall b \in B.\, (a, b) \in R \vee (b, c) \in S\}$ and cells α as above exist iff the converse implication $\big(f(a), g(c)\big) \in S \Rightarrow (a, c) \in R$ holds. $\mathbb{R}\mathrm{el}^\bullet$ is also an equipment, with companions and conjoints being given by complements of graphs of functions and their opposites, respectively, and its underlying fibration is (up to equivalence) the dual fibration of $Sub(Set)$. These "dual" notions of composing relations also appear in the study of linear logic via linear bicategories, and we expect dual equipments to connect to linear logic as well. This talk will also address some logical properties of $\mathbb{R}\mathrm{el}^\bullet$, and the properties of $\mathbb{R}\mathrm{el}$ that induce them.

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A Categorical Framework for Coherence Theorems

Nelson Niu

Joint work with: Jonathan L. Rubin

Categories with coherently associative, commutative, and distributive products encapsulate higher algebraic structures throughout mathematics: for instance, in homotopy theory, they are the inputs to May's infinite loop space machines [6, 7], Segal's K-theory [10], and multifunctorial, multiplicative, and/or equivariant analogues of these by Elmendorf–Mandell [1], Guillou–May–Merling–Osorno [2, 3, 4], and Yau [11]. In ongoing work, we establish a general categorical approach to proving Mac Lane-like coherence theorems versatile enough to incorporate (weak) distributivity laws, module and algebra categories, bicategories, and the higher arity twisted products that appear in equivariant settings. Building on Mac Lane's original proof of his coherence theorem for (symmetric) monoidal categories [5] and Rubin's coherence theorem for his equivariant normed symmetric monoidal categories [8], we employ tools from combinatorics, logic, and rewriting theory such as Newman's Diamond Lemma [9] to solve categorical normalization problems on the universal parameter categories representing categorical structures of interest. Our approach clarifies the necessary coherence axioms and invariants. We aim to leverage our work to simplify the characterization of bimonoidal categorical input to Yau's multifunctorial equivariant algebraic K-theory.

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A Functorial Weak Factorisation System from Path Types

Noah D. Ortiz

Joint work with: Yiming Xu

We show that the path types structure proposed by Awodey and Hua [2] gives rise to a *functorial* weak factorisation system (WFS). Our work draws on *path object factorisations* as in [7, 3, 1] and from Gambino and Garner's *identity type weak factorisation system* on the syntactic category of dependent type theory [4]. In analogy to the latter, we define our left maps as the maps that left-lift against a single map t, representing a *universe*. Our right maps are then the *saturation* of t (i.e. the right-lifts against the left maps). We arrive at the same characterisation of left maps as [4, 7]: The left maps are exactly the *strong deformation retracts*. In contrast, however, to both, our construction of the factorisation is *functorial*.

Awodey and Hua [2] give a very general setting in which we can interpret intensional identity types of Martin-Löf type theory: The setting is a category with finite limits, an exponentiable interval $1 \rightrightarrows I$, and a universe $\mathsf{t} : \dot{\mathsf{T}} \to \mathsf{T}$ that is a *normal Hurewicz fibration* admitting a pullback to its own path type $\dot{\mathsf{T}}^I \to \dot{\mathsf{T}} \times_{\mathsf{T}} \dot{\mathsf{T}}$. In such a category, we generate our WFS from the *type families* (pullbacks of t), by taking the left maps to be left-lifts against the type families, and restricting to the subcategory of *types* (those objects whose maps to 1 are type families).

Our construction relies crucially on the fact that each type admits a *normal connection* $X^I \to (X^I)^I$ as constructed in [2], regarded as a strong deformation retraction of the path type X^I onto the type X. Following [1, p. 82] and [7, Proposition 6.1.4], we give each map $Y \xrightarrow{f} X$ a path object factorisation $Y \to P_f \to X$. This is a variation on Gambino and Garner's factorisation in [4] through an "identity context". Unlike their setting, where the factorisation need not be functorial [4, Remark 12], our factorisation through P_f is functorial.

The history of the path object factorisation can be traced back to van den Berg and Garner [7] in a finite limit category, with an abstract *path object* on an object X playing the role of X^I as in our work. While they assume an internal category structure on the interval, we do not; and while we assume a universe, they do not. Later, van den Berg and Moerdijk [3] show that the factorisation can be performed without requiring the category to have all pullbacks. They do not yield a weak factorisation system because, in their setting, diagonal fillers commute merely up to homotopy. In both our work and in [7], the left maps are the strong deformation retracts. We therefore expect our construction to also be *cloven*, as in their setting, or even algebraic [6, 5].

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Lazy Categories

Robert Paré

Strict double categories are category objects in **Cat**, and much of their basic properties follow from the general theory of category objects, where the results are proved using standard, albeit complicated, limit arguments. When proving such results we secretly think that we are working with sets. Now, **Cat** has many set-like properties but it is not **Set**, so we shouldn't be surprised when some things don't work. For example profunctors don't compose properly.

Instead of pretending that categories are sets, we turn this on its head and pretend that sets are categories. Well, not sets but the next best thing, the objects of a nice topos. Following Verity we embed **Cat** in a topos which closely resembles it, whose objects we call lazy categories.

We investigate how much category theory carries over and what are the advantages of doing this.

Twisted double functors and their applications

Evan Patterson

Joint work with: Michael Lambert, David Jaz Myers

In a seminal paper on Yoneda theory for double categories [1], Paré begins by noticing that lax double functors are the right kind of mapping to generalize the Hom functor from categories to double categories. Specifically, the *Hom functor* on a double category $\mathbb{D}$ is a lax double functor $\mathrm{Hom}_{\mathbb{D}} : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathbb{S}\mathsf{et}$ that sends a pair of objects x and y in $\mathbb{D}$ to the set of tight morphisms $f : x \to y$ between them and acts with a pair of tight morphisms by pre- and post-composition. Alternatively, the Hom functor can be described as a *normal* lax double functor $\mathrm{Hom}_{\mathbb{D}} : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathbb{C}\mathsf{at}$ and a Yoneda theory developed along these lines [2].

Properly speaking, this is a *tight* Yoneda theory for double categories. What, then, could be a *loose* Yoneda theory? In attempting to construct a loose Hom functor, an obstacle is immediately encountered: the loose Hom should send a pair of objects x and y to the set (or category) of loose morphisms $m : x \nrightarrow y$ and act with a pair of loose morphisms by pre- and post-composition. But that is generally not possible when the receiving double category is $\mathbb{S}\mathsf{et}$ or $\mathbb{C}\mathsf{at}$ since it requires sending loose morphisms to tight morphisms (functions or functors). It is only possible in the special case that $\mathbb{D}$ is a *strict* double category, so that the transpose $\mathbb{D}^{\top}$ makes sense.

We develop a theory of *twisted double functors*, a new kind of double functor that sends loose morphisms to tight morphisms and vice versa. That is less straightforward than it might initially seem since twisted functors must allow composition comparisons in both the loose-to-tight and tight-to-loose directions, even though are former are usually invertible. Our prime example is the *twisted Hom functor* on a double category $\mathbb{D}$, a twisted normal lax functor denoted $\mathbb{D}(-,=) : \mathbb{D}^{\mathrm{co}} \times \mathbb{D} \looparrowright \mathbb{C}\mathsf{at}$. From this example many others can be constructed by precomposition with ordinary double functors, particularly the *twisted representables* $\mathbb{D}(a,-) : \mathbb{D} \looparrowright \mathbb{C}\mathsf{at}$ for each object $a \in \mathbb{D}$. Twisted representables encompass the familiar (bi-)indexed categories; for example, when C is a category with finite limits, the twisted representable $\mathbb{S}\mathsf{pan}(\mathsf{C})(1,-) : \mathbb{S}\mathsf{pan}(\mathsf{C}) \looparrowright \mathbb{C}\mathsf{at}$ contains the canonical self-indexing of C.

Having defined twisted double functors, as well as their morphisms and higher morphisms, we introduce *loosely discrete double opfibrations* as a notion of discrete opfibration internal to **Cat**. Our main result is an elements construction establishing, for any double category $\mathbb{D}$, an equivalence between *twisted copresheaves* on $\mathbb{D}$, defined to be twisted normal lax functors $\mathbb{D} \looparrowright \mathbb{C}\mathsf{at}$, and loosely discrete opfibrations over $\mathbb{D}$. We also introduce a collage construction establishing an equivalence between *twisted bimodules*, defined to be twisted normal lax functors $\mathbb{E}^{\mathrm{co}} \times \mathbb{D} \looparrowright \mathbb{C}\mathsf{at}$, and the *double barrels* (aka *loose bimodules*) recently proposed as a key concept of double-operadic systems theory [3]. As a result, we obtain several equivalent descriptions of what we believe to be a significant new concept in double category theory.

The foregoing is work in progress that is nearly complete. In future projects, we hope to utilize the concepts introduced here to develop a loose Yoneda theory for double categories, as well as to axiomatize *compact double categories* from the perspective of adjunctions between twisted Homs.

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Presheaves on Markov Categories and Expectation Values

Paolo Perrone

Joint work with: Tobias Fritz

Categorical probability is the study of the structural aspects of probability, statistics and related fields by means of category-theoretic methods. Two categorical structures abound in the field:

1. Via *probability monads* [1, 4], one models probabilistic maps as morphisms in Kleisli categories. The structure of algebra of a probability monad amounts to equipping the underlying space with a notion of forming expectation values. Therefore algebras are interesting because expectation values are among the most commonly used and fruitful ideas in probability theory.

2. *Markov categories* [3] can be seen as an axiomatization of categories of probabilistic maps, including but not limited to the Kleisli categories of probability monads. Via their monoidal structure, they allow one to talk about stochastic independence, conditioning, sufficient statistics, etc.

In a plain Markov category, there is no general and simple way to talk about expectation values. In this work [2], we aim to address this shortcoming.

Idea 1. *Given a Markov category* C *and a presheaf* $\Phi : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$, *we think of elements* $\phi \in \Phi(X)$ *as "functions on* X*" and of the functoriality of* Φ *as the formation of pointwise expectation values.*

For example on $\mathsf{FinStoch}$, the category of finite sets and stochastic matrices, there is a presheaf with $\Phi(X) = \mathbb{R}^X$, and where the action on morphisms is given by

$$\Phi(f)(\phi)(x) \coloneqq \sum_{y \in Y} f(x|y)\, \phi(y).$$

When f is deterministic, this reduces to the usual precomposition of functions. When the domain of f is singleton, then this is precisely the usual notion of expectation value of ϕ with respect to f.

We will argue that this results in a general framework for expectation values that is better behaved than algebras of a probability monad.[1] Thus Idea 1 broadens the scope of categorical probability to certain *quantitative* aspects, such as conditional expectations, variance and covariance, and several inequalities, bringing the theory of categorical probability one step closer to the mathematical structures and methods used by practitioners. We expect that it opens up the possibility of developing many further aspects of traditional probability theory in the categorical setting.

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[3] Tobias Fritz, *A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics.* Advances in Mathematics 370 (2020), available at arXiv:1908.07021.

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[1] For example, with Stoch being the category of measurable spaces and Markov kernels, for every $n \in \mathbb{N}$ the hom-functors $\mathsf{Stoch}(\_, [-n, n])$ for $n \in \mathbb{N}$ are presheaves that model bounded measurable functions and their expectation values. The filtered colimit of these presheaves in the functor category $[\mathsf{Stoch}^{\mathrm{op}}, \mathsf{Set}]$ is the presheaf of all bounded measurable functions, which plays a central role in traditional probability theory. But in the category of algebras of the Giry monad, the filtered colimit of the intervals $[-n, n]$ is trivial, i.e. the singleton space.

Elementary ∞-Toposes from Type Theory

Maximilian Petrowitsch

Joint work with: Daniël Apol

Since its conception, it has been speculated that Homotopy Type Theory (HoTT) [7] is the internal language of particular higher categories also called *elementary ∞-toposes*. A precise formulation of this statement was given by Kapulkin and Lumsdaine in [3, Conj. 3.7] and is known as the *internal language conjecture*. Here, HoTT is understood as Martin-Löf dependent type theory with dependent sums, dependent products, intensional identity types and univalent universes.

The conjecture fits into a series of important steps that have been made so far towards establishing such a connection between higher categories and type theory. Kapulkin and Szumiło [4] showed that dependent type theory with intensional identity types is the internal language of finitely complete ∞-categories. Assuming in addition dependent products, Kapulkin [2] proved that every model presents a locally cartesian closed ∞-category. Shulman [6] showed that HoTT can be interpreted as internal language into any Grothendieck ∞-topos.

Proving a correspondence between HoTT and elementary ∞-toposes is currently an open problem. Elementary ∞-toposes are supposed to generalise both Grothendieck ∞-toposes and ordinary 1-toposes. There is currently no generally agreed upon definition of an elementary ∞-topos, even though there has been increasing interest in the topic in the last decade. We define an elementary ∞-topos as a finitely complete, locally cartesian closed ∞-category with enough univalent morphisms. In its precise statement, the internal language conjecture then asserts that there is a Dwyer-Kan equivalence $\mathrm{Cl}_\infty\colon \mathbf{CxlCat}_{\mathtt{HoTT}} \xrightarrow{\sim} \mathbf{ElTop}_\infty$ between the category of categorical models of HoTT and the category of elementary ∞-toposes, induced by sending each model to its ∞-localisation at the class of homotopy equivalences. Proving that this functor exists, i.e. that the ∞-localisation takes values in $\mathbf{ElTop}_\infty$, and showing that this is a Dwyer-Kan equivalence has so far been an open problem.

In the talk, we will present the work [5] in which we prove the existence of this functor Cl_∞, a first step towards proving the conjecture. That is, we show that every model of HoTT presents an elementary ∞-topos via its ∞-localisation. First, we use the fact that every model of HoTT has the structure of a tribe in the sense of Joyal [1]. We extend Joyal's theory of tribes by introducing the notion of a univalent fibration in a tribe and the notion of a univalent tribe and we show that every categorical model of HoTT is such a univalent tribe. Then, we prove that every univalent tribe presents via its ∞-localisation an elementary ∞-topos inducing finite colimits and a subobject classifier under the presence of pushout types and propositional resizing. Thus, the functor Cl_∞ can be obtained as a composite: $\mathbf{CxlCat}_{\mathtt{HoTT}} \to \mathbf{UnivTrb} \to \mathbf{ElTop}_\infty$.

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On the Distributive Law in Cartesian Multicategories

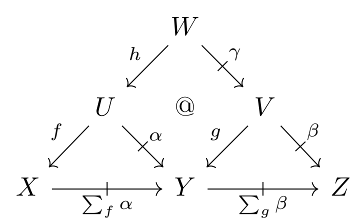
Claudio Pisani

Cartesian operads can be seen as a way to encode algebraic theories; though they are essentially equivalent to Lawvere theories, the perspective is different. Cartesian multicategories (or cartesian colored operads) correspond to the many sorted theories. In this talk we present cartesian multicategories as algebras for a natural monad **cart** on symmetric multicategories, and show how this notion is in several respects conceptually advantageous. This analysis is carried out in the framework of *unbiased* symmetric multicategories [1] where, as is natural in the symmetric context, sequences are replaced by families.

In fact, unbiased symmetric multicategories are simply sum-preserving double functors $\mathbb{M} \to \mathbb{P}\mathrm{b}$ (to the double category of pullback squares in finite sets) which are discrete fibrations in the tight direction. The loose arrows of $\mathbb{M}$ should be thought of as families of arrows in the multicategory, indexed by the underlying functions in $\mathbb{P}\mathrm{b}$, while tight arrows and cells give reindexing. The classical axioms are in this way naturally embodied by the double categorical structure. This approach, which removes the awkwardness arising from an unnatural skeletal indexing, turns out to be rather effective and opens up new perspectives. In particular, it allows for a base sensitive study of multicategories and renders transparent the links with Joyal's species.

For instance, *plain* multicategories are sum-preserving double discrete fibrations $\mathbb{M} \to \mathbb{T}\mathrm{ot}$, where $\mathbb{T}\mathrm{ot}$ is (the double categorical form of) the multiplicative species of total orders. Furthermore, unbiased symmetric monoidal categories arise when $\mathbb{M} \to \mathbb{P}\mathrm{b}$ is, in the loose direction, an opfibration, while the algebras for a symmetric multicategory $\mathbb{M} \to \mathbb{P}\mathrm{b}$ are morphisms $\mathbb{A} \to \mathbb{M}$ which are *discrete* opfibrations, and so are themselves symmetric multicategories.

Symmetric operads can be seen as *multiplicative* species of structures and cartesian operads are obtained by endowing them also with a *sum*, such that multiplication-composition *distributes* over sums. In fact, we define cartesian multicategories as the algebras for the monad which takes a symmetric multicategory $\mathbb{M}$ to the multicategory $\mathbf{cart}\,\mathbb{M}$ whose loose arrows are spans (actually, "enhanced" spans [1]) formed by a tight arrow f and a loose arrow α in $\mathbb{M}$; these are composed in the usual way for spans, except that we use $\mathbb{M}$-cells in place of pullbacks. Thus, a cartesian structure $\sum : \mathbf{cart}\,\mathbb{M} \to \mathbb{M}$ provides a way to evaluate spans, giving the "sum" $\sum_f \alpha$ of α along f. The functoriality of $\sum$ on loose arrows says that the sum of a composition of spans is the same as the composition of their sums: $\sum_g \beta \sum_f \alpha = \sum_{fh} \beta\gamma$,



Since cells in $\mathbb{M}$ (like @) are defined by reindexing (which, for non-injective functions, duplicate elements), this can be indeed seen as a kind of generalized distributive law, holding in any cartesian multicategory.

If R is a monoid, we have the "cocartesian" operad $\mathbb{R}_\triangleright$ associated to the multiplicative species R^I of labellings in R: a structure over a set I is a family $\alpha : I \to R$. Cartesian structures on $\mathbb{R}_\triangleright$ correspond to rig structures on R, and the distributive law therein is of course the usual one.

On the other hand, for any set S, the species S^{S^I} gives the endomorphism operad $\mathbb{E}\mathrm{nd}(S)$ which is also cartesian: sums are given by duplication-deletion of variables in the multivariate functions $\alpha : S^I \to S$, and the distributive law says, for instance, that the two ways of calculating $\beta(\alpha(x);\alpha(x))$ are equivalent: we can either duplicate the value $\alpha(x)$ and evaluate β at it, or duplicate the function α itself, compose it with β and evaluate it at the duplicate of x. Cartesian morphisms $\mathbb{R}_\triangleright \to \mathbb{E}\mathrm{nd}(S)$ amount to modules S on R, and can themselves be considered as cartesian multicategories over $\mathbb{R}_\triangleright$; their loose arrows over $(r_i)_{i\in I}$ are "combinations" $(r_i s_i)_{i\in I}$ and sums reduce them on equal s_i.

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Weak ∞-categories via fat Delta

Stiéphen Pradal

Understanding higher categories boils down to understanding their coherence complexity. In a nutshell, coherences are equations between operations such as composition and units, and may be strict (genuine equality) or weak (up to higher homotopies). Simplicial methods provide some of the most effective tools for modelling these objects. However, when these techniques are used to encode higher categorical structures, the degeneracy relations enforce strict coherence conditions on units. While often harmless, this rigidity becomes problematic in contexts where weak unit structures are essential, notably in the study of the cobordism ∞-category [5] and in Simpson's conjectures [1]. To address this issue, J. Kock [2] introduced the category $\underline{\Delta}$, named fat Delta, as a variant of the simplex category that allows degeneracies to be treated up to homotopy, at the cost of a localisation at the class of *vertical* morphisms. In this work, we develop a model of weak ∞-categories based on $\underline{\Delta}$-spaces, analogous to the role played by complete Segal spaces [3] in the simplicial setting. Pursuing J. Kock's original idea of using $\underline{\Delta}$ to approach Simpson's conjectures, we aim to bring new insights to the study of these conjectures and more generally to the theory of weak ∞-categories.

We introduce the notion of *fat ∞-category*, defined as simplicial presheaf over $\underline{\Delta}$, a.k.a. $\underline{\Delta}$-space, satisfying appropriate vertical, Segal, and completeness conditions. The vertical condition plays a central role: it allows degeneracy data, and in particular units, to be reconstructed up to homotopy, therefore yielding weak unit structures. These conditions are extracted as left Bousfield localisations of the Reedy model structure on $\underline{\Delta}$-spaces with respect to the Kan–Quillen model structure on simplicial sets. In particular, we can show the following.

Proposition 1. *There is a simplicial, left proper and combinatorial model structure on $\underline{\Delta}$-spaces such that the cofibrations are monomorphisms and the fibrant objects are fat ∞-categories.*

We further explore alternative characterisations of the conditions. By adapting the notions of *horn* and *saturation* [6] to the $\underline{\Delta}$-space context, we provide a reformulation of the vertical and Segal conditions in terms of horns, and of the completeness in terms of saturation.

Finally, we discuss ongoing work comparing fat ∞-categories with other established models of weak ∞-categories. In particular, we outline a conjectural Quillen equivalence with Harpaz's model of quasi-unital ∞-categories [4]. This model is based on marked semisimplicial spaces, which allows us to form a natural functor by assigning to each $\underline{\Delta}$-space its underlying semisimplicial space and space of markings. Conversely, by appropriately using the *f-degenerate marked semisimplex* introduced in [4], one can construct the functor in the opposite direction.

Conjecture 2. *The functors induce a Quillen equivalence between the model categories encoding quasi-unital ∞-categories and fat ∞-categories.*

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Categorical structures for Gödel incompleteness and Löb's theorem

Sridhar Ramesh

In provability logic, a key principle is Löb's theorem, stating that if the provability of P provably entails P, then P itself is provable (in modal logic notation, $\Box P \vdash P$ has as a consequence $\vdash P$). This was first discovered in the follow-up work on Gödel's incompleteness theorems, with Gödel's results viewable as a special case of Löb's theorem. This same formal pattern of Löb's theorem was later observed to also arise in several other seemingly unrelated contexts. For example, the formal pattern of Löb's theorem is also of note in modal logic as corresponding to well-founded transitive Kripke frames. And the formal pattern of Löb's theorem also describes certain fixed point constructions studied under the name of "guarded recursion" in programming language theory.

In the author's dissertation [4], currently being prepared for publication, it is observed that a particularly simple class of category-theoretic structures serve as an abstract environment for deriving Löb's theorem and its associated fixed point constructions, allowing for a vastly generalized and unified understanding of the scope of applicability of these, including all of the above contexts. These are the structures we call "introspective theories".

Specifically, we define an "introspective theory" to be a category with finite limits T with an internal category with finite limits C, along with a natural transformation from the self-indexing of T to C (construed as contravariant functors from T to the category of categories with finite limits).

Remarkably, this simple structure is in itself enough to derive Löb's theorem, as well as guarded recursion at both the term and type level. We also demonstrate how this abstraction offers a clean unification of the interpretation of the Gödel-Löb incompleteness theorems in traditional logic or via arithmetic universes a la Joyal (as in [2]), along with the interpretation by Birkedal et al of guarded recursion in presheaves over well-founded orders (as in [1]), along with the distinct classical interpretation of the GL modal logic in well-founded transitive Kripke frames.

This significantly extends the connection between Gödel's incompleteness theorem and category theory observed by Lawvere in [3], by now giving a category-theoretic account of the the Gödel coding process itself. As this account is not tied to the specific context of natural number arithmetic, it also allows us to furthermore observe uncountable and uncomputable structures in which the Gödel incompleteness phenomena still arise.

We also explore free instances of our structure, which turn out to admit a tractable explicit description. The free introspective theory is called in the author's dissertation "the theory of geminal categories", and there are a number of further illuminating relationships discovered between introspective theories and geminal categories.

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Homotopy Pullbacks and Homotopy Groups in CAT

Daniel Ramras

In their work on algebraic quantum field theory, Roberts, Ruzzi, and Vasselli gave a purely combinatorial description for the exact sequence of fundamental groups associated to certain fibered functors between posets (*poset net bundles*). This relies on a description of homotopy in terms of zig-zags of morphisms, similar to work of Evrard from the 1970s.

The work discussed in this talk grew out of a desire to give a combinatorial description for the entire long exact sequence in homotopy. To this end, we describe a functor from cospans in Cat to Cat, which models the homotopy pullback after passing to geometric realizations. This functor is a categorical version of the usual description of homotopy pullbacks in Top, but with geometric paths replaced by zig-zags of morphisms (of arbitrary length). As such, our result for homotopy pullbacks is an infinitary version of Barwick–Kan's Theorem B_n. In the special case of homotopy fibers, we recover a result of Shoikhet (simplifying an earlier model for homotopy fibers due to Evrard); this model is an infinitary version of Dwyer–Hirschhorn–Kan's Theorem B_n (itself a variation on Quillen's Theorem B).

Applying our model for homotopy pullbacks to a cospan

$$\{C\} \hookrightarrow \mathcal{C} \hookleftarrow \{C\},$$

we recover the categorical model for based loop spaces introduced by Evrard, from which one can extract Evrard's model for homotopy groups in terms of n–dimensional zig-zag diagrams in the underlying category. Our approach allows for a combinatorial description of the boundary map in the long-exact Mayer–Vietoris sequence associated to an arbitrary homotopy pullback; in the setting of algebraic quantum field theory, this yields the desired combinatorial description for the long exact sequence of a poset net bundle.

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From Filter Quotient Model Categories to Type Theory

Nima Rasekh

Categorical logic is an area of category theory that uses categorical methods to construct and study models of various mathematical foundations, such as set theories and type theories. The idea is to construct categories whose objects axiomatically behave like sets or types, and then use the categorical structure to interpret the logical operations and axioms of the chosen foundation. We can use this approach to construct both standard and non-standard models of various foundations. One elegant way to construct non-standard models is via *filter quotients* of categories. It has, in particular, been used to construct models of set theory where the continuum hypothesis fails [1].

Here, for a given category $\mathcal{C}$ and a suitable poset of subterminal objects Φ in $\mathcal{C}$, the *filter quotient* $\mathcal{C}_\Phi$ is given as a filtered colimit $\mathrm{colim}_{U \in \Phi^{op}} \mathcal{C}_{/U}$. This construction in particular comes with a projection functor $P_\Phi \colon \mathcal{C} \to \mathcal{C}_\Phi$, which preserves much of the structure of $\mathcal{C}$, hence preserving the property of being a model of a given foundation, resulting in new models.

In recent decades, we have seen the rise of homotopical foundations for mathematics, and in particular, various *homotopy type theories (HoTT)*. Analogously, we have also witnessed the development of ∞-categorical models of homotopy theory, proving that the ∞-category of spaces and of sheaves on spaces are indeed models of HoTT. However, the study of non-standard models of HoTT via ∞-categorical filter quotients has remained largely unexplored. Here, for a given ∞-category $\mathcal{C}$ and a suitable poset Φ, the ∞-categorical filter quotient is defined analogously as a filtered colimit of slice ∞-categories $\mathcal{C}_\Phi = \mathrm{colim}_{U \in \Phi^{op}} \mathcal{C}_{/U}$, which again comes with the projection functor $P_\Phi \colon \mathcal{C} \to \mathcal{C}_\Phi$ [2].

Constructing models of HoTT via ∞-categories requires overcoming major technical challenges. Due to their syntactic nature, type theories are very strict, whereas ∞-categories are inherently weak structures. One effective way to bridge this gap is to leverage *model categories*. Indeed, model categories are strict categories that nonetheless come with an *underlying ∞-category*.

This approach breaks down the task into a two-step process. First, one constructs a suitable model category for a chosen ∞-category. Then, one shows that the model category satisfies the conditions necessary to model HoTT. In the case of the ∞-category of spaces, this has been realized via the Kan model structure on simplicial sets, whereas for ∞-categories of sheaves (Grothendieck ∞-topoi) this goal has been achieved via *type-theoretic model topoi* [3].

In this talk, I will show how this two-step approach can be extended to filter quotient ∞-categories, resulting in new non-standard models of HoTT. More specifically, I show that many filter quotient ∞-categories admit *filter quotient model categories* [4]. Moreover, I demonstrate that the projection functor P_Φ preserves the property of being a model for HoTT, resulting in new non-standard models of HoTT [5, 6]. These non-standard models exhibit intriguing behaviors, such as non-standard natural numbers, that enrich our understanding of homotopical foundations.

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Torsion theories as short exact sequences

Ülo Reimaa

Joint work with: Graham Manuell, Nelson-Martins Ferreira, Fosco Loregian

A torsion theory allows one to decompose objects of a category into pairs of objects falling into two fixed subcategories. We will make the point that torsion theories can be seen as short exact sequences of categories and explore how this idea works with different notions of torsion theory [1, 3].

To illustrate the point, consider the following notion of torsion theory on a category $\mathcal{A}$. Equip $\mathcal{A}$ with full subcategories $\mathcal{T}$ and $\mathcal{F}$, along with an object Z that lies in both subcategories, such that:

- there exists precisely one arrow $T \to F$ whenever $T \in \mathcal{T}$ and $F \in \mathcal{F}$;
- for every object $A \in \mathcal{A}$, there exists a square (1) that is bicartesian, meaning it is simultaneously a pullback and a pushout, with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

$$\begin{array}{ccc} T & \longrightarrow & Z \\ \downarrow & (1) & \downarrow \\ A & \longrightarrow & F \end{array}$$

The interpretation is that the square (1) exhibits the decomposition of an object A into an (essentially unique) “short exact sequence” $T \to A \to F$. Indeed, in the special case where Z is the zero object, asking the square (1) to be a pullback is the same as requiring $(T \to A) \cong \mathsf{ker}(A \to F)$ and asking the square (1) to be a pushout is the same as requiring $(A \to F) \cong \mathsf{coker}(T \to A)$. In general, Z need not be the zero object, although it can be thought of as lying between the initial object and the terminal object.

It turns out that $\mathcal{T}$ is coreflective and $\mathcal{F}$ is reflective in $\mathcal{A}$, with the arrows $T \to A$ and $A \to F$ in the square (1) giving the coreflection into $\mathcal{T}$ and reflection into $\mathcal{F}$. Therefore, our data builds the diagram

$$\mathcal{T} \underset{\longleftarrow}{\overset{\hookrightarrow}{\perp}} \mathcal{A} \underset{\hookleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{F}, \tag{2}$$

which can be viewed as a short exact sequence, in the sense that the composite $\mathcal{T} \underset{\hookleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{F}$ is the zero adjunction (the essentially unique adjunction that factors through the terminal category), in addition to

$$\Big(\mathcal{T} \underset{\longleftarrow}{\overset{\hookrightarrow}{\perp}} \mathcal{A}\Big) \simeq \mathsf{ker}\Big(\mathcal{A} \underset{\hookleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{F}\Big) \quad \text{and} \quad \Big(\mathcal{A} \underset{\hookleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{F}\Big) \simeq \mathsf{coker}\Big(\mathcal{T} \underset{\longleftarrow}{\overset{\hookrightarrow}{\perp}} \mathcal{A}\Big),$$

in a sense that is appropriate for adjunctions. The main observation here is the following.

Observation. Exact sequences of the form (2) are essentially the same as torsion theories on $\mathcal{A}$.

Given a short exact sequence of categories in the above sense, the central category $\mathcal{A}$ can be viewed as a category of extensions with kernel coming from $\mathcal{T}$ and cokernel coming from $\mathcal{F}$. This idea can also be used to generate a cofree torsion theory on a pair of full subcategories of a category.

Example. If the categories are pointed then Z is necessarily the zero object and one recovers torsion theories in the classical sense.

Example. A large class of non-pointed examples is provided by Artin glueings of toposes [2]. The open subtopos and its closed complement form a torsion theory, with Z the corresponding subterminal.

We can interpret (2) as a split extension if the functor $\mathcal{A} \to \mathcal{F}$ additionally has a left adjoint. Such “split extensions” admit a nice theory of semi-direct products, which is especially well-behaved if the categories in question are protomodular. For instance, when dealing with opposites of toposes or semi-abelian categories.

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A toolbox for weighted (∞, n)-limits

Martina Rovelli

Joint work with: Lyne Moser, Nima Rasekh

An (∞, n)-category consists of objects and an $(\infty, n-1)$-category of higher morphisms between any two objects, while an $(\infty, 0)$-category is simply a space. In this setting, many constructions of interest have the universal property of the limit of an (∞, n)-functor $F\colon \mathcal{J} \to \mathcal{C}$, possibly weighted by an (∞, n)-functor $W\colon \mathcal{J} \to \mathcal{C}at_{(\infty,n-1)}$.

Developing a solid theory of limits and colimits in the (∞, n)-categorical setting for $n > 0$ has been a central goal of my research program with Moser and Rasekh. In earlier work [1, 2], we proposed and validated a definition of weighted limits and colimits in this setting, and in this talk we will report on current developments of the theory [3].

We will primarily discuss completeness results and provide explicit formulas for weighted (co)limits under specific assumptions on $\mathcal{C}$. These include:

- a formula for weighted limits in $\mathcal{C}$ in terms of products, (co)tensors, limits over the simplex category, under the assumption that $\mathcal{C}$ admits these elementary (co)limits;
- a formula for weighted limits in $\mathcal{C}$ in terms of the Grothendieck constructions $\int_{\mathcal{J}} F$ and $\int_{\mathcal{J}} W$ of F and W, when $\mathcal{C}$ is $\mathcal{C}at_{(\infty,n-1)}$;
- a pointwise formula for weighted limits in $\mathcal{C}$, when $\mathcal{C}$ is an (∞, n)-category of functors valued in a suitably complete (∞, n)-category.

If time permits, we will also indicate how several classical theorems on limits and colimits extend to the (∞, n)-categorical setting. These include:

- a generalization of Fubini's formula, describing the commutation of weighted (∞, n)-limits with weighted (∞, n)-limits;
- a characterization of weighted cofinality for (∞, n)-functors;
- a cancellation property for comma objects in an (∞, n)-category; and
- a construction of the free completion of an (∞, n)-category under weighted limits.

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Soberness as idempotent-completeness:
towards a formal model theory of virtual ultracategories

Gabriel Saadia

Joint work with: Errol Yuksel

The notion of idempotent-complete (or Cauchy complete) category arises from completeness of metric spaces. In this talk, we honor the topological roots of this notion and reinterpret it in the setting of *virtual ultracategories*, a categorification of topological spaces defined independently in [1, 2, 3] last summer.

Virtual ultracategories (or v·u·categories for short) are a multicategorical generalization of ultracategories, abstracting the structure of "categorified convergence of ultrafilters" on the category of points of a topos. The key motivating theorem is that any topos with enough points can be reconstructed as the category of sheaves over its v·u·category of points. Hence, v·u·categories are strong enough to encode any topos with enough points. This raises the following question: when is a v·u·category *sober* (i.e. arises as the v·u·category of points of a topos)? In this talk, we focus on the following restricted case:

Question. *When is a full sub-v·u·category of a sober v·u·category sober? Or from a logical point of view, when is a class of models of a geometric theory, geometrically axiomatizable?*

More precisely, for a subclass $\mathcal{K} \subseteq \mathcal{M}$ of the v·u·category $\mathcal{M}$ of points of a topos, the *soberification* of $\mathcal{K}$ yields an extension $\mathcal{K} \subseteq \check{\mathcal{K}} \subseteq \mathcal{M}$ known as *subclosure* in the localic case [4]. The following result shows that this subclosure corresponds exactly to closure under *v·u·retracts*, a straightforward analogue of the usual categorical notion of retraction in the setting of v·u·categories. This result strongly echoes Lawvere's result [5] relating completeness of metric spaces with idempotent-completeness.

Theorem. *A point belongs to $\check{\mathcal{K}}$ if and only if it is a v·u·retract of points in $\mathcal{K}$. In particular, $\mathcal{K}$ is sober (or equivalently, geometrically axiomatizable) if and only if $\mathcal{K}$ is v·u·idempotent-complete, and the class of points $\mathcal{K}$ is separating if and only if any point is a v·u·retract of points in $\mathcal{K}$.*

In other words, the geometric notion of *soberness*, the logical notion of *geometrically axiomatizable class* of models, and the algebraic notion of *v·u·idempotent-completeness* all coincide.

Finally, it is worth mentioning that the notion of v·u·retract is powerful enough to recover the standard notion of filtered colimit, which plays a central role in the theory of accessible categories. We thus view the notion of v·u·idempotent-completeness as a first step toward a more general understanding of accessibility for v·u·categories, opening the way to a formal model theoretical study of these structures.

[1] S. van Gool and J. Marquès and U. Tarantino, *Toposes with enough points as categories of étale spaces*, preprint arXiv:2508.09604, 2025.

[2] A. Hamad, *Generalised ultracategories and conceptual completeness of geometric logic*, preprint arXiv:2507.07922, 2025.

[3] G. Saadia, *Extending conceptual completeness via virtual ultracategories*, preprint arXiv:2506.23935, 2025.

[4] P. Johnstone, *Open maps of toposes.* Manuscripta Mathematica, 1980.

[5] W. Lawvere, *Metric spaces, generalized logic and closed categories.* Rend. Sem. Mat. Fis. Milano, 1973.

Linear higher rewriting and applications to diagrammatic algebras

Léo Schelstraete

Rewriting theory is the study of normal forms in structures presented by generators and relations. It has roots in the word problem for groups, and in the linear case, to the resolution of polynomial equations (so-called Gröbner bases). In this talk, I will discuss how to extend the (linear) theory higher: to (linear) monoidal categories, presented by generators and relations using string diagrams. This builds on the theory of polygraphs, see e.g. [2]. Emphasizes will be put on how various categorical constructions naturally appear, out of necessity—this includes semistrict categories, double categories, and spans.

Our approach is driven by examples, and in particular by diagrammatic algebras as they appear in representation theory and low-dimensional topology (e.g. Kac–Moody 2-categories or Hecke categories): we will give an explicit application to a conjecture in these fields. This talk also aims at introducing these problems to category theorists, and explain why a categorical viewpoint is becoming increasingly necessary. If time permits, we will discuss future development, such as implementation or ∞-refinement.

This talk is based on [1].

[1] L. Schelstraete, *Rewriting modulo in diagrammatic algebras and application to categorification*, preprint arXiv:2502.03028, 2025.

[2] D. Ara et al., *Polygraphs: from rewriting to higher categories*, London Mathematical Society Lecture Note Series, 495, Cambridge Univ. Press, Cambridge, 2025; MR4866320

Associated bundles in restriction category world

Florian Schwarz

In classical differential geometry, principal bundles are locally the cartesian product of a manifold and a group, vector bundles are locally the cartesian product of a manifold and a vector space. It is a classical result that there is a one to one correspondance between the category of principal GL_n-bundles and the category of vector bundles with fiber $\mathbb{R}^n$.

In [1] Cockett and Cruttwell generalized vector bundles as differential bundles in tangent categories. In [2], Cockett and I generalized principal bundles to restriction categories and in particular to tangent restriction categories.

Does the correspondance between principal and vector bundles hold up in this generalized setup?

It does not. I will give an example of a restriction category in which there are more differential bundles than principal bundles. However one direction of the correspondance still holds. The construction that sends a principal bundle to a vector bundle is known as the associated bundle. We will perform this construction in the setting of tangent restriction categories, obtain a functor and describe its properties. This does not just recover a classical result from differential geometry, it also provides a strategy for constructing examples of differential bundles in tangent restriction categories.

[1] J. R. B. Cockett and G. S. H. Cruttwell, *Differential structure, tangent structure, and SDG*, Applied Categorical Structures 22 (2014), 331–417.

[2] R. Cockett and F. Schwarz, *Lie groups in tangent join restriction categories*, preprint arXiv:2509.18410,2025

Coalgebraic-modal extensions of doctrines

José Siqueira

Joint work with: David Jaz Myers (work in progress)

The modelling of phenomena is an iterative process; one's understanding of something and the very scope of interest evolves over time. This poses a challenge to logicians: if you settle on a specific logic to describe an object, you may soon find it inappropriate.

This is notably the case in the study of *systems*. On one hand, the language itself may need to be expanded if new features and behaviours are to be studied (e.g., add *modalities* to form new predicates), and on another hand you may wish to change the flavour of system under consideration (therefore changing the logical *contexts* and the underlying type theory). Often, both procedures are necessary. For example, one may wish to go from having Boolean predicates about machine states to having *temporal* predicates about *streams* of states.

This talk will address the following main question: when is one procedure compatible with the other in such a way that our starting logic can be "upgraded" to a modal one on the new type of system? More formally, given a basic hyperdoctrine $P\colon \mathcal{C}^{\mathbf{op}} \to \mathcal{D}$, a comonad F on $\mathcal{C}$, and a monad L on $\mathcal{D}$, we are concerned with determining the lax monad morphisms $F^{\mathbf{op}} \to L$ with underlying 1-morphism P for which the induced map of Eilenberg-Moore categories is again a hyperdoctrine.

This is best studied by working with double categories instead, as (regular) hyperdoctrines can be equivalently described as certain symmetric monoidal double pseudofunctors $\mathbb{S}pan(\mathcal{C})^{\mathbf{op}} \to \mathbb{Q}t(\mathcal{D})$ [1], and the flavours of systems themselves can be expressed usefully in double-categorical language [2]. In this light, we will introduce a 2-category $\mathcal{D}oct$ of double pseudofunctors between cartesian double categories and consider its Eilenberg-Moore objects. We will discuss when the logical extension scenarios we are interested in correspond to monoidal pseudomonads in $\mathcal{D}oct$, so that their Eilenberg-Moore objects provide the desired coalgebraic-modal extensions of the starting logic. If time permits, I will also comment on how this approach relates to traditional coalgebraic modal logic [3].

[1] J. Siqueira, *Double-functorial representation of regular monoidal structures*, arXiv:2508.06637, 2025.

[2] Sophie Libkind, David Jaz Myers, Towards a double operadic theory of systems, arXiv:2505.18329, 2025

[3] Clemens Kupke, Dirk Pattinson, Coalgebraic semantics of modal logics: An overview, Theoretical Computer Science, Volume 412, Issue 38, pp. 5070–5094, 2011.

Categories by Kan extension

David Spivak

Categories can be identified—up to isomorphism—with polynomial comonads on **Set**. The left Kan extension of a functor along itself is always a comonad—called the density comonad—so it defines a category when its carrier is polynomial. We provide a number of generalizations of this to produce new categories from old, as well as from distributive laws of monads over comonads. For example, all Lawvere theories, all product completions of small categories, and the simplicial indexing category $\mathbf{\Delta}^{\mathrm{op}}$ arise in this way. Along the way, we will see various constructions of non-polynomial comonads as well.

Strictification of $(\infty, 1)$-Categories.

Kimball Strong

∞-groupoids are infinite-dimensional generalizations of groupoids in which all axioms (associativity, identity, etc.) hold only "up to (coherent) homotopy." This flexibility makes them powerful enough to encode homotopy types, but much too complicated to reason with in any algebraic manner. ω-groupoids are a "strict" version, in which all the relevant axioms hold on the nose, rather than up to homotopy. There is an adjunction

$$\infty\mathrm{Gpd} \underset{\iota}{\overset{\mathrm{St}}{\rightleftarrows}} \omega\mathrm{Gpd}$$

In which the left adjoint St "strictifies" an ∞-groupoid, forcing axioms to hold strictly rather than simply up-to-homotopy. Since ∞-groupoids are equivalent to homotopy types, St gives an algebraic invariant for spaces. It turns out that ω-groupoids are essentially a blend of chain complexes and (ordinary) groupoids, and the functor St refines simultaneously the homology and fundamental groupoid of a space. Various homological theorems that hold only for simply connected spaces can be expanded to all spaces by working with ω-groupoids rather than chain complexes: for instance, the functor St reflects weak equivalences, which specializes to simply connected spaces as the statement that a map $f : X \to Y$ is a weak equivalence of simply connected spaces iff it is a homology equivalence.

In this talk, we will discuss the construction of an analagous invariant for $(\infty, 1)$-categories: we construct a Quillen adjunction

$$(\infty,1)\mathrm{Cat} \underset{\iota_1}{\overset{\mathrm{St}_1}{\rightleftarrows}} (\omega,1)\mathrm{Cat}$$

Where $(\omega, 1)$Cat is the category of $(\omega, 1)$-categories, which are "strict" versions of $(\infty, 1)$-categories [2]. We show that while essentially a mix of chain complexes and (strict) $(2, 1)$-categories, this invariant is fairly strong: in particular, we show that St_1 reflects weak equivalences, thus giving a homological/categorical mechanism to test for equivalences of $(\infty, 1)$-categories [3].

As an important technical step, we prove a change-of-base theorem for monoidal model categories: given a Quillen adjunction $L : \mathcal{V} \rightleftarrows \mathcal{W} : R$ with R lax monoidal, we prove that (under some model categorical assumptions) there is an induced Quillen adjunction

$$\mathcal{V}\text{-}\mathrm{Cat} \underset{R}{\overset{L^{\mathrm{Cat}}}{\rightleftarrows}} \mathcal{W}\text{-}\mathrm{Cat}$$

Notably, we improve on previous results of this form (e.g. [1]) by not requiring that L be "weak Quillen monoidal," the homotopical analog of strong monoidality [3].

[1] Fernando Muro, *Dwyer-Kan homotopy theory of enriched categories*, Journal of Topology **8** (2012)

[2] Kimball Strong, *An Enriched Approach to the Strictification of $(\infty, 1)$-Categories*, preprint arXiv:2510.04254, 2025.

[3] Kimball Strong, *Strictifications of Higher Categories*, Ph.D. Thesis, 2026.

Picard Infinity Groupoids with Underlying Globular Sets

Johnathon Taylor

The *Stable Homotopy Hypothesis*, or *SHH* for short, states that the homotopy category of Picard n-groupoids equipped with categorical equivalences is equivalent to the homotopy category of stable homotopy n-types equipped with stable homotopy equivalences for all $n \geq 0$ (see [3]). In essence, the SHH is a statement about how two different categories, both of whose objects are obtained by weakening the point-set equality and axioms for an abelian group, have equivalent homotopy theories. Moreover, the SHH is motivated by the Homotopy Hypothesis, the Baez-Dolan Stablization Hypothesis [2], the Freudenthal Suspension Theorem (see Section 11.2 of [6]), and Thomason's Theorem [9].

On one end of the conjectured equivalence in the SHH, we have the stable homotopy types, i.e. connective spectra and their n-truncated models for all $n \geq 0$. Connective spectra are well-studied and several models for connective spectra exist, such as the infinite loop spaces of [4] and the Segal spaces of [8]. On the other end, we have Picard ∞-groupoids and their n-truncated models for all $n \geq 0$. There exist simplicial and topological models where the Stable Homotopy Hypothesis is known to be true, such as group-like E_∞-spaces [5] and the Picard–Tamsamani model [7]. On the other hand, we would like a model for Picard n-groupoid with an underlying globular set where the SHH is true; however, there is no notion of Picard n-groupoid past dimension 2 in the literature with an underlying globular set.

In this talk, we provide the first globular model for Picard ∞-groupoids, which have an underlying Grothendieck ∞-groupoid (see [1]), as models over a specially constructed limit sketch. Since these Picard ∞-groupoids have an underlying Grothendieck ∞-groupoid, they have an underlying globular set. Then Picard n-groupoids are merely defined to be Picard ∞-groupoids whose underlying Grothendieck ∞-groupoids are n-truncated. Our main goal in this talk is to provide the tools and construction strategy used to define the limit sketch whose models are defined to be Picard ∞-groupoids.

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[2] J. C. Baez and J. Dolan, Higher-dimensional algebra and topological quantum field theory, J. Math. Phys. **36** (1995), no. 11, 6073–6105; MR1355899

[3] N. Gurski, N. Johnson and A. M. Osorno, The 2-dimensional stable homotopy hypothesis, J. Pure Appl. Algebra **223** (2019), no. 10, 4348–4383; MR3958095

[4] J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics, Vol. 271, Springer, Berlin-New York, 1972; MR0420610

[5] J. P. May, E_∞ spaces, group completions, and permutative categories, in *New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972)*, pp. 61–93, London Math. Soc. Lecture Note Ser., No. 11, Cambridge Univ. Press, London-New York, ; MR0339152

[6] J. P. May, *A concise course in algebraic topology*, Chicago Lectures in Mathematics, Univ. Chicago Press, Chicago, IL, 1999; MR1702278

[7] L. Moser et al., Stable homotopy hypothesis in the Tamsamani model, Topology Appl. **316** (2022), Paper No. 108106, 40 pp.; MR4438951

[8] G. B. Segal, Categories and cohomology theories, Topology **13** (1974), 293–312; MR0353298

[9] R. W. Thomason, Symmetric monoidal categories model all connective spectra, Theory Appl. Categ. **1** (1995), No. 5, 78–118; MR1337494

Transfinite iteration of functors as an extensional reflection

Paul Taylor

Infinite or transfinite iteration of functors is the principal way in which mainstream mathematics goes outside the logic of an elementary topos. It can be encoded using von Neumann's class recursion theorem, but in 100 years the set theorists have failed to explain the two main components of this to the wider community. One is Replacement and the other is using impredicative higher order logic to derive recursion from induction.

Category theory is a mature and powerful discipline that should be able to do this in its native language, adopting the ideas without the dogma of other subjects. My proposal is to use extensional reflection of well founded coalgebras, in which "extensionality" is defined using a general factorisation system. This was shown to be *consistent* with ZF in a recent ItaCa lecture.

As a new *axiom* this is intended as an addition to those for an elementary topos, or maybe other foundational systems, but also as a new *tool* in our toolbox. One application is to recover ZF, which first requires the von Neumann hierarchy, *i.e.* the transfinite iteration of the covariant powerset, but full recovery involves other issues about ZF that distract from the category theory.

Thirty years after renouncing excluded middle and working on ordinals in category theory, I am not going to revert to the classical ones. However, the various constructive ones are not nicely behaved, for example it is not clear what diagram *shapes* are required to form colimits over them.

Extensional well founded coalgebras for general functors come to the rescue here, because they form (class) *preorders* rather than categories and behave very like sets under inclusion. Also the iterates of a functor are extensional and well founded, so transfinite *colimits* behave like set-theoretic *unions*.

Pullbacks give the $\mathcal{M}$-maps of a factorisation system in any arrow category. Applied to (op)fibrations they can express equations. Thus transfinite iteration of functors may be obtained as an extensional reflection in a category of opfibrations of (small) posets.

The papers and lecture slides are at

`www.paultaylor.eu/ordinals/`

including a new historical study of *Old and new proofs of the order-theoretic fixed point theorem.*

Isoregular theories, accessible 2-categories, and free constructions

Giacomo Tendas

Joint work with: Nicola Gambino

The importance of free constructions in 2-dimensional category theory is widely recognised; it is enough to think for instance about free completions under limits or colimits of some shape, as well as free regular and exact completions. A way to show that such free constructions exist is usually provided by an adjoint functor theorem, an instance of which is proved in [1] in the context of accessible 2-categories with flexible limits.

The purpose of this talk is to present a notion of 2-dimensional theory, in the sense of logic, whose 2-categories of models are indeed accessible with flexible limits, and for which the forgetful 2-functors that naturally arise between them are accessible and flexible-limit preserving. This, together with the result of [1], implies that free constructions, in form of left biadjoints, always exist in such a framework.

We call the 2-dimensional theories in question *isoregular*. The idea being that, just like in ordinary cartesian logic one is allowed to express properties defined by unique existential quantification, within *isoregular* logic one can express properties defined by an existence which is unique up to (unique) isomorphism.

Examples of 2-categories arising this way include those whose objects are: Categories with (co)limits of some shape, Grothendieck fibrations, Clans, Regular and Exact categories, as well as Protomodular, and Semiabilian categories.

[1] J. Bourke, S. Lack and L. Vokřínek, *Adjoint functor theorems for homotopically enriched categories*, Advances in Mathematics 412:108812, 2023.

Why we should pay more attention to Blass' theorem

Christopher Townsend

It is well known that a natural numbers object is needed in our base topos if we want to have classifying toposes for geometric theories. But even back in 1978 the question was posed by Johnstone and Wraith as to whether the condition was necessary; that is, does the existence of classifying toposes for geometric theories imply that our base topos must have a natural numbers object? (See the remark before Prop 4.9 in [JW78].) Since having a natural numbers object is essentially the same thing as satisfying the Axiom of Infinity, answering the question positively allows us to consider infinity in a way that is very much removed from its usual statement.

About a decade later Blass [B89] answered the question positively and B4.2.11 of the Elephant [J02] contains an excellent account showing that this observation is easy to derive using the direct image (back to the base topos) of the generic object in the classifying topos of the relevant theory.

In this talk I will outline the above in more detail and then provide some comments as to how it might be possible to understand infinity using these methods as a purely topological construct, removing it from its usual 'discrete' setting. The key is to re-interpret Blass's theorem as a statement about localic groupoids, using the Joyal and Tierney result, [JT84]. I will explain that there are two different ways of viewing infinity in this manner; they are dual to one another in the sense that discrete can be considered to be dual to compact Hausdorff. Time permitting I will explain a conjectured third way of isolating infinity topologically that is self-dual.

The talk will lean on joint work with Henry, [HT23].

[B89] Blass, A. *Classifying topoi and the axiom of infinity*, Algebra Universalis **26**, (1989) 341-345.

[HT23] Henry, S. and Townsend, C.F. *A classifying groupoid for compact Hausdorff locales*. Preprint, 2023. https://arxiv.org/abs/2310.07785

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[JT84] Joyal, A. and Tierney, M. *An Extension of the Galois Theory of Grothendieck*, Memoirs of the American Mathematical Society **309**, 1984.

Advances in the Unification of Localic and Realizability Toposes

Davide Trotta

Joint work with: Maria Emilia Maietti

Localic and realizability toposes are two central classes of toposes in categorical logic. The main difference between these two families is that realizability toposes are the main examples of elementary toposes which are not Grothendieck toposes.

In order to investigate their common features, Hyland, Johnstone and Pitts introduced the tripos-to-topos construction in [1], that is a construction producing a topos starting from a suitable kind of Lawvere doctrine called tripos, and showed that both localic and realizability toposes are instances of this construction. This result was the first fundamental step to unify the treatment of these two classes of toposes.

The main purpose of this talk is to carry on this line, by focusing on the geometric properties of localic toposes and on how to generalize them to triposes and toposes obtained from them.

In detail, we show that every topos obtained from a tripos whose base category has weak dependent products and a generic proof is a topos of j-sheaves with respect to a suitable Lawvere-Tierney topology over a topos abstracting the sheafification and, respectively, the notion of localic presheaf category [7].

We prove that this topos of abstract presheaves is precisely the topos obtained by applying the $\exists$-completion [8] and then the tripos-to-topos construction to the starting tripos, and that the abstract sheafification arises from a suitable adjunction between the starting tripos and its $\exists$-completion.

Relevant examples include all the toposes obtained from triposes whose base category is Set, such as: localic toposes, the Effective topos, the Modified Realizability topos [2], the Extensional Realizability topos [3], the Dialectica topos [4], the Krivine topos [5]. As a further significant example of a topos obtained from a tripos that is not Set-based, we mention the topos of extended Weihrauch degrees recently introduced in [6].

Then we show that the tripos-to-topos and the $\exists$-completion constructions can be combined to produce a tower of toposes, where each topos is a topos of j-sheaves for the following one, from a given tripos. The abstract topos of presheaves turns out to be the second step of this tower.

Finally, we show how our analysis is flexible enough to be generalized to the context of predicative toposes to include, e.g. the predicative toposes obtained by Martin-Löf's type theory or Homotopy Type theory.

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[2] J.M.E. Hyland, C.H.L. Ong, *Modified realizability toposes and strong normalization proofs*, in: Bezem, M., Groote, J.F. (Eds.), Typed Lambda Calculi and Applications, Springer Berlin Heidelberg, Berlin, Heidelberg (1993), 179–194.

[3] J. van Oosten, *Extensional realizability*, Annals of Pure and Applied Logic **84** (1997), 317–349.

[4] B. Biering, *Dialectica Interpretations-A Categorical Analysis*, PhD Thesis, University of Copenhagen (2008).

[5] T. Streicher, *Krivine's classical realisability from a categorical perspective*, Mathematical Structures in Computer Science **23** (2013), 1234–1256.

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[7] M.E. Maietti and D. Trotta, *An Algebraic Abstraction of the Localic Sheafification via the Tripos-to-Topos Construction*, preprint (2025), `https://arxiv.org/pdf/2511.06945`.

[8] M.E. Maietti and D. Trotta, *A characterization of generalized existential completions*, Annals of Pure Applied Logic **174**-103234 (2023), 37.

Relative differential categories, differential clones and Fermat theories

Jean-Baptiste Vienney

A *differential category* [2] is a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ enriched over commutative monoids, together with a monad S on $\mathcal{C}$ and a differentiation operator $\mathrm{d}\colon SA \to SA \otimes A$ satisfying standard axioms. Relative monads are a generalization of monads where the underlying functor is not necessarily an endofunctor [1]. Our main contribution in this talk is the introduction of the notion of a *relative differential category* where the monad is replaced by a relative monad. Every differential category is a relative differential category but the converse is false. An example is Vec_k with its usual tensor product and with $S(n) = k[x_1, \ldots, x_n]$. Another one is $\mathsf{Vec}_\mathbb{R}$ with its usual tensor product and with $S(n) = C^\infty(\mathbb{R}^n, \mathbb{R})$. This framework allows one to work directly with polynomials and smooth functions, instead of using coordinate-free constructions such as the symmetric algebra.

These two examples can be generalized using clones [3]. A rig clone $\mathcal{O}$ is given by a commutative rig $\mathcal{O}(n)$ for every $n \geq 0$ together with projections and composition operations satisfying familiar identities. We introduce the notion of a *differential clone* as a rig clone together with partial derivative operations $\partial_i\colon \mathcal{O}(n) \to \mathcal{O}(n)$. Differential clones are equivalent to the differential theories of [2]. For every differential clone, the symmetric monoidal category $(\mathrm{Mod}_{\mathcal{O}(0)}, \otimes, \mathcal{O}(0))$ is a relative differential category. In particular, we recover our two previous examples of relative differential categories by choosing $\mathcal{O}(n) = k[x_1, \ldots, x_n]$ or $\mathcal{O}(n) = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$.

A *Fermat theory* [4] is a ring clone where differentiation is axiomatized algebraically through difference quotients. We show that every Fermat theory is a differential clone. The differential clones $\mathcal{O}(n) = k[x_1, \ldots, x_n]$ and $\mathcal{O}(n) = C^\infty(\mathbb{R}^n, \mathbb{R})$ are Fermat theories. However, not every differential clone is a Fermat theory and we provide a counterexample inspired from differential Galois theory [5]. Finally, given any topological field k whose topology is nontrivial and nondiscrete, we define smooth functions from k^n to k following [6]. We show that $\mathcal{O}(n) = C^\infty(k^n, k)$ gives a Fermat theory. This provides the first connection between smooth functions on topological fields and differential categories.

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On the Category of Graded Monads

Victoria Vollmer

Joint work with: Marco Paviotti, Dominic Orchard

Monads have become an invaluable tool in both pure mathematics and computer science. However, in many practical applications, we encounter situations where monads need to carry additional structure or parameters, necessitating a generalization known as *graded monads*. Graded monads allow us to refine the coarse-grained structure of classical monads by decorating them with grades from an indexing monoid, enabling finer control over composition and interaction of effects.

To develop a robust formal theory for graded monads, we follow the approach that proved successful for classical monads: we seek a 2-categorical perspective that reveals their underlying structure. Bénabou defined a monad as a morphism from the terminal bicategory [1]. Street capitalized on the structure of 2Cat and considered the lax-functor category $[1, \kappa]_{lax}$ as the 2-category of monads [2]. This way of mapping any 2-category, κ, to the 2-category of monads over κ defines the functor $\mathsf{Mnd} : \mathsf{2Cat} \to \mathsf{2Cat}$. Street's formal treatment of monads allowed for the wider adoption of monads for a multitude of uses in both mathematics and computing. Orchard et al. made the key conceptual leap by generalizing Bénabou's definition of a monad to graded monads, replacing the terminal bicategory with the delooping of a monoid [3]. This generalization provides the natural jumping-off point for a formal treatment of graded monads: just as Street's perspective provides a 2-categorical lingua franca for monads, we provide the same for graded monads.

We follow Street and Bénabou, defining a 2-functor $\mathsf{Gmd} : \mathsf{MonCat} \times \mathsf{2Cat} \to \mathsf{2Cat}$ that takes a monoidal category I, and a 2-category κ, to the lax-functor category $[BI^{op}, \kappa]_{lax}$. We show Gmd is a graded monad on 2Cat, which identifies distributive laws as graded monads in the category of graded monads themselves and gives us a notion of composition for graded monads, $\mathsf{comp}_{I,J}(\kappa) : \mathsf{Gmd}(J, \mathsf{Gmd}(I, \kappa)) \to \mathsf{Gmd}(J \times I, \kappa)$. We include a dual notion for graded comonads, $\mathsf{Gcmd}(\kappa) = [BI^{op}, \kappa]_{oplax}$ and equivalences for distributive laws for graded monads and graded comonads in the style of Power and Watanabe [5], which contrast with the graded distributive laws of Gaboardi et al. [4].

With this machinery in place, we can close some additional open questions, namely, "what is the free graded monad?" We present a free-forgetful adjunction between the category of I^*-graded monads over κ and I-indexed endofunctors over κ, showing that the *free graded monad* is given by the the left Kan extension of the unit of the free monoid monad. We observe that the category of I-graded monads, $\mathsf{Gmd}(I, \kappa)$ is still insufficient for understanding other relationships between graded monads: the fixed monoid I means that this category does not capture reindexing and re-grading. To solve this we define the category of graded monads as the 2-Grothendieck construction of $\mathsf{Gmd}(-, \kappa)$ for a fixed 2-category κ. This construction gives us the appropriate 2-category with graded monads of any index.

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Towards Tangent Structures for Orbifolds by Way of Tangent Categories

Geoff Vooys

Joint work with: Dorette Pronk

Orbifolds are geometric objects which are of interest in algebraic topology, differential topology, representation theory, certain flavours of differential geometry, and even in music theory (cf. [7] for an introduction and a discussion of the etymology of "orbifold"). An orbifold, broadly speaking, can be seen as a smooth manifold which is locally folded and glued along some local symmetries together with appropriate symmetry-preserving transition functions. As a result, orbifolds give ways to study smooth differential objects and the ways they interact with smooth symmetries and natural locations in which to study equivariant cohomology (of both Borel or Bredon flavour; cf. [3] for Borel cohomology for orbifolds and [4], [2] for Bredon cohomology for orbifolds). However, a difficulty with studying the tangential information encoded on an orbifold lies in the fact that defining the tangent tangent vectors which appropriately respect the symmetries along which the orbifold is glued require not mere tangent vectors, but instead require tangent vectors together with equivariant descent-theoretic information showing that these vectors are suitably symmetry-stable. In this talk we will propose how to define such tangent-theoretic information by using the language of tangent categories, a semantic tool for studying structural differential geometry discovered in [6] and [1], and pseudolimits of tangent categories.

In [4] we showed how to build pseudolimits in the 2-category of tangent categories as well as how to use them to give descent-equivariant tangent structures for smooth G-spaces (for Lie groups G acting smoothly on smooth manifolds). In this talk we will extend the work of [5] and illustrate how to use the tangent structures for smooth G-spaces to define tangent structures on orbifolds as follows. Because the charts in orbifold atlases are open subspaces of $U \subseteq \mathbb{R}^n$ equipped with a smooth action of a finite group S, we first equip each chart U with its S-equivariant tangent structure and then take the pseudolimit of such tangent categories in order to arrive with a tangent structure for our given orbifold.

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The ∞-category of ∞-categories in simplicial type theory

Jonathan Weinberger

Joint work with: Daniel Gratzer, Ulrik Buchholtz

Simplicial type theory (STT) was introduced by Riehl and Shulman [1] to leverage homotopy type theory to prove results about $(\infty, 1)$-categories. Initial work on simplicial type theory focused on "formal" arguments in higher category theory and, in particular, no non-trivial examples of ∞-category theory were constructible within STT. More recent work has changed this state of affairs by applying techniques developed initially for cubical type theory to construct the ∞-category of spaces. In [2], we complete this process by constructing the ∞-category of ∞-categories, recovering one of the main foundational results of ∞-category theory (straightening–unstraightening) purely type-theoretically. STT extends HoTT with a *directed interval*, a postulated totally ordered lattice $(\mathbb{I}, 0, 1, \leq)$. This new type is meant to represent the category $0 \to 1$—an interpretation justified by the model of STT in simplicial spaces—and we then use $\mathbb{I}$ to define morphisms in an arbitrary type A as ordinary functions $\mathbb{I} \to A$ constraining the endpoints: $\hom_A(a, b) = \sum_{f:\mathbb{I}\to A} f\,0 = a \times f\,1 = b$. [1] then demonstrate that the definition of an ∞-category can be formulated concisely as a *predicate* isCat on types, essentially requiring every pair of composable morphisms have a unique composite. Furthermore, they show that ordinary functions between such types constitute functors and that other classical definitions in ∞-category theory become expressible. Subsequent work has further expanded this approach, developing fibered category theory, (co)limits, etc. As an extension of HoTT, STT comes equipped with a (hierarchy of) universes and it is therefore natural to ask: *Is $\mathcal{U}$ a recognizable category, e.g., the category of categories?*

Unfortunately, the answer is negative; $\mathcal{U}$ is the canonical example of a type that is *not* an ∞-category in STT. In fact, even if one considers simple subtypes of the universe (e.g., $\sum_{A:\mathcal{U}} \mathsf{isCat}(A)$) one does *not* obtain a category, as synthetic morphisms $\mathbb{I} \to \sum_{A:\mathcal{U}} \mathsf{isCat}(A)$ neither compose nor faithfully represent functors. However, it has long been conjectured that the category of categories should be constructible in STT as a certain subtype of the universe. We address this final gap in the foundations of STT by settling this conjecture affirmatively and constructing the category of categories as a subtype $\mathrm{Cat} \hookrightarrow \mathcal{U}$ and verifying its essential properties:

1. Cat is the base of the universal cocartesian fibration
2. Cat is *directed univalent*, i.e., for global elements A, B in Cat we have $\hom_{\mathrm{Cat}}(A, B) \simeq (A \to B)^{\simeq}$.
3. Cat is a category (i.e., Segal and Rezk) and simplicial.

We promote this to a straightening–unstraightening theorem à la [5] and give first applications such as defining monoidal ∞-categories, K-theory of monoidal 1-categories, and using Cat's *structure homomorphism principle* to compute the morphisms of the lax slice $1 \downarrow \mathrm{Cat}$ and of the category of marked categories.

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Reconstructing the canonical extension in the stable setting

Joshua Wrigley

Joint work with: Sam van Gool

Reconstruction problems. Since the category of models does not determine a first-order theory up to any typical notion of equivalence, e.g. Morita equivalence, it has been a persistent theme in categorical logic to identify what extra structure is needed to obtain a "reconstruction theorem". Examples in the literature include *ultrastructure* à la Makkai [6], or the data of a *topological category* [1, 8].

Canonical extensions. In this talk, we will discuss a reconstruction problem for Coumans' *first-order canonical extension* [2], which has seen a recent renewal of interest, for instance in [3] (alongside a resurgence in reconstruction theorems more generally, e.g. [7]). Coumans' construction takes the *coherent doctrine* $D_{\mathbb{T}}$ associated with a theory $\mathbb{T}$, à la Lawvere [4], and applies the *canonical extension* $(-)^{\delta}$ from lattice theory; this can be understood as introducing a new predicate for each (model-theoretic) *type* of $\mathbb{T}$. Indeed, if $\mathbb{T}$ is a classical theory (and assuming the axiom of choice), $D_{\mathbb{T}}^{\delta}$ consists of the powerset of types in each context.

Our contribution. We will present the following:

- Firstly, we give a streamlined proof that the *complete points* of the canonical extension $D_{\mathbb{T}}^{\delta}$ correspond precisely to the countably saturated models of $\mathbb{T}$ (which is only alluded to implicitly in previous literature, cf. [5, §1.4]);
- Secondly, we show that if $\mathbb{T}$ is *countably stable* and in a countable language, the canonical extension $D_{\mathbb{T}}^{\delta}$ can be reconstructed from merely knowing the underlying sets of the countable saturated models; in other words, given countably stable theories $\mathbb{T}_1, \mathbb{T}_2$, there is a natural isomorphism $D_{\mathbb{T}_1}^{\delta} \cong D_{\mathbb{T}_2}^{\delta}$ if and only if the countable saturated models of $\mathbb{T}_1$ and $\mathbb{T}_2$ are equivalent as categories *over* **Set**.

The above results will be the subject of a submission to a forthcoming volume of the *Outstanding Contributions to Logic* series in honour of Hilary Priestley.

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Monotone-light factorizations, very-well-behaved epireflections and categories of models of sketches

João J. Xarez

The genesis of this work is in [1]. Its main results show how to obtain the monotone-light factorization from n-categories via n-preorders ($n \geq 1$), using sketches of presheaves, without resorting to some more complicated tools or $\mathcal{V}$-categories (as in [5] and [6]). There are also some other new results here (cf. [7]), some as displayed below. The provided theoretical framework will hopefully be applied to new settings, as the algebraic one for instance.

Firstly, precise conditions on how to obtain *very-well-behaved* epireflections are explored and improved from the author's previous papers [3] and [4]; meaning that, beginning with a monad and a prefactorization system on a category, is produced a reflection with stable units (stronger than semi-left-exactness, also called admissibility in categorical Galois Theory) and an associated monotone-light factorization.

Secondly, as a first application of the conditions above, deriving from adjunctions given by left Kan extensions for presheaves, we will show that, for a pseudo-filtered category $\mathbb{J}$ in which every arrow is a monomorphism, the colimit functor on $Set^{\mathbb{J}}$ produces a *very-well-behaved* epireflection; astonishingly, in the very simple case with $\mathbb{J} = \mathbf{2}$, the monotone-light factorization is non-trivial.

Thirdly and more importantly, new results are presented that grant *very-well-behaved* subreflections from the very-well-behaved reflections induced by an adjunction given by right Kan extensions for presheaves. These subreflections are obtained by restricting to the models of a sketch; it is showed finally that the known *very-well-behaved* reflection of n-categories into n-preorders is an example of this process (being n any positive integer).

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Craig Interpolation for Subgeometric Logics

Lingyuan Ye

Joint work with: Ivan Di Liberti

In 1957 Craig proved his interpolation theorem. Roughly speaking, it means if there is a deduction between two first-order formulas $\varphi \vdash \psi$, then one can find an interpolant θ, which belongs to the *common* language of φ, ψ, that $\varphi \vdash \theta$ and $\theta \vdash \psi$. Since then, interpolation proved to be extremely influential in several areas of logic. Among the most prominent variations of Craig's theorem, Pitts' works [3, 4] delivered an algebraically flavoured version of Craig interpolation for first-order intuitionistic logic as well as coherent logic, from a categorical logic perspective. The definition of interpolation in categorical logic terms is a generalisation of the original interpolation in its syntactic form, and also of the one used in algebraic logic, which primarily applies to propositional fragments.

In our paper [1], we explore a cluster of fragments of *geometric logic* and assess to what extent these fragments admit a form of Craig interpolation theorem. The main difference of our approach is that, instead of trying to prove interpolation for *specific* fragment of logic, we establish *uniformly* a Craig-interpolation-type theorem for a wide class of subfragments of geometric logic.

In our investigation, we have identified a property of logic that plays a key role in establishing interpolation results. In the language of doctrine (a.k.a lax-idempotent monads on Lex), the property states that it should *preserve slicing*. This implies the 2-category of algebras are closed under taking slices, which is indeed such a fundamental property of doctrines associated with logic and type theory that perhaps has not been paid enough attention to in the literature. We provide a classification of the interpolation property for doctrines preserving slicing as an *exactness property*. In particular, we introduce a notion of *t-conservative maps* of lex categories, and show that it belongs to an orthogonal factorisation system for any *finitary* doctrine on Lex. Using this, we are able to provide a classification as below:

Theorem 1. *Let T be a finitary doctrine on lex categories preserving slicing. It has the interpolation property iff t-conservative maps are closed under cocomma in $\mathsf{Alg}(\mathsf{T})$.*

For a working notion of *fragment* of geometric logic, we follow the authors' previous work [2]. From *loc. cit.* every logic $\mathcal{H}$ induces a doctrine $\mathsf{T}^{\mathcal{H}}$, and the existence of an *étale map classifier* for the logic $\mathcal{H}$ is tightly connected to the associated doctrine $\mathsf{T}^{\mathcal{H}}$ preserving slicing:

Proposition 1. *If a fragment $\mathcal{H}$ has an étale map classifier, then $\mathsf{T}^{\mathcal{H}}$ preserves slicing.*

Using the techniques presented in [2], especially the *classifying topos* construction for $\mathcal{H}$, we are able to obtain our main theorem:

Theorem 2. *Let $\mathcal{H}$ be a fragment of geometric logic between the regular and coherent fragment having an étale map classifier. Then $\mathcal{H}$ has the interpolation property.*

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Effective codescent morphisms in varieties of universal algebras with the amalgamation property

Dali Zangurashvili

The notion of an effective descent morphism is one of the main notions of A. Grothendieck's descent theory. The problem of characterizing such morphisms (called a descent problem) in varieties of universal algebras is simple and well-known. In contrast to this case, for the dual category of a variety of universal algebras, very little is known. The first result of A. Grothendieck, still used in commutative algebra and algebraic geometry, can be seen as a partial solution to the descent problem for the dual category of commutative rings. The complete solution was independently obtained by A. Joyal and M. Tierney, and B. Mesablishvili later. This talk is devoted to the descent problem in general dual-algebraic categories, which was posed by G. Janelidze. We deal with the case of categories with the amalgamation property (which is satisfied by a number of well known categories, but not by that of commutative rings). In [3], we gave the criterion for a morphism of a category with pushouts and equalizers to be a codescent morphism (we use the term "codescent" for descent in dual categories). In [4], we reduced the codescent problem to the simpler one, replacing arbitrary pushouts in codescent data by pushouts of monomorphisms. Applying this, we obtained the characterization of effective codescent morphisms in a number of categories of topological nature: topological spaces, compact Hausdorff topological spaces, normal topological spaces, Banach spaces, and some others. Moreover, we found the sufficient condition formulated in syntactical form for all codescent morphism of a variety of universal algebras to be effective [5]; with its aid the complete answer to the codescent problem in the categories of groups, loops, quasigroups was given. In [1], we employed the tools of the term rewriting systems theory to the codescent problem, and solved the codescent problem for the varieties of Mal'tsev algebras, idempotent quasigroups, unipotent quasigroups, and some others. Recently, appying the results of this paper, we solved the codescent problem in the varieties of n-loops and n-quasigroups [6], and ternary rings [2]. Note that a ternary ring is not a ring in the traditional sense. However, the category of ternary rings contains the category of traditional rings with unit as a full reflective subcategory. Our results imply that the class of morphisms between commutative rings which are effective codescent in the category of ternary rings coincides with that of monomorphisms satisfying the traditional ring-theoretic ideal extension property. Note that the problem whether the latter class of monomorphisms coincides with that of monomorphisms satisfying the above-mentioned Joyal-Tierney's condition was of interest for certain purely ring-theoretic reasons in the past century, and the negative answer was given independently by several authors. The author gratefully acknowledges the financial support of Shota Rustaveli National Science Foundation of Georgia (FR-24-8249).

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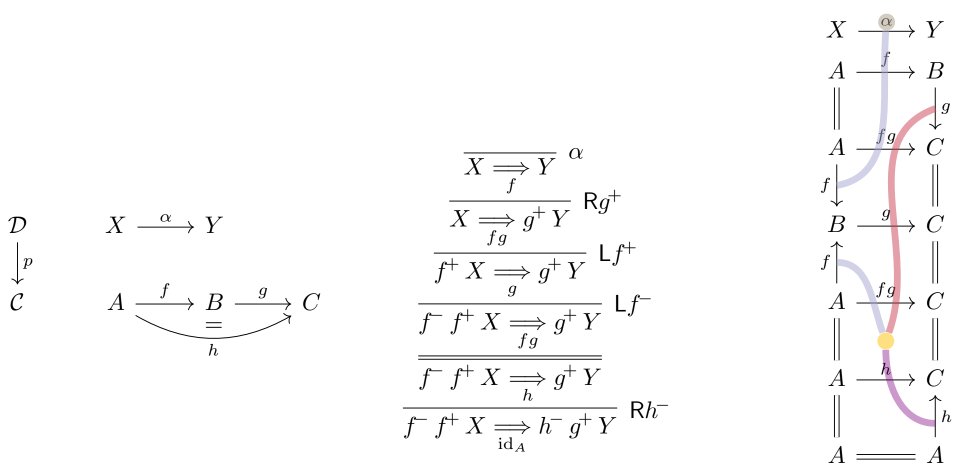
The free bifibration on a functor

Noam Zeilberger

Joint work with: Bryce Clarke, Gabriel Scherer

A bifibration is a functor $p : \mathcal{D} \to \mathcal{C}$ which is both a fibration and an opfibration; intuitively, objects in the domain $\mathcal{D}$ can be pushed or pulled along arrows of the codomain $\mathcal{C}$. Fibrations, opfibrations, and bifibrations play an important role throughout category theory and computer science, particularly in categorical logic. While the construction of the free (op)fibration via comma categories is well-known, the problem of building free bifibrations has remained relatively unexplored until now, with the only explicit construction appearing in unpublished work of Lamarche [3] – although the problem is also closely related to a problem considered by Dawson, Paré, and Pronk, of "freely adjoining adjoints" to a category [2].

In this work, we develop a novel proof-theoretic construction of the free bifibration $\Lambda_p : \mathcal{B}\mathrm{if}(p) \to \mathcal{C}$ in which objects of $\mathcal{B}\mathrm{if}(p)$ are formulas of a primitive "bifibrational logic", and arrows are derivations in a cut-free sequent calculus modulo a notion of permutation equivalence (see middle of figure below, with the free bifibration generated from the data of the functor p on the left). Remarkably, instantiating the construction to the identity functor generates a *zigzag double category* $\mathbb{Z}(\mathcal{C})$, which coincides with the free double category with companions and conjoints (or fibrant double category) on $\mathcal{C}$. This, in turn, suggests a natural string diagram calculus for morphisms in the domain of a free bifibration (see right side of figure).



The approach adapts smoothly to the more general task of freely adding pushforwards and pullbacks relative to some restricted classes of arrows $\mathcal{P}$ and $\mathcal{N}$, recovering so-called *ambifibrations* as a special case when $(\mathcal{P}, \mathcal{N})$ form a factorization system. Ideas from proof theory guide us through a series of progressively stronger normal forms, deriving a canonicity result under assumption that the base category is factorization preordered relative to $\mathcal{P}$ and $\mathcal{N}$. Finally, we identify a number of surprising and interesting examples. These include a category of plane trees generated as a free bifibration over ω, and a category of increasing forests generated as a free ambifibration over Δ, which contains the lattices of noncrossing partitions as quotients of its fibers by the Beck-Chevalley condition for bicartesian squares.

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